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M. E. Ash

Generation of Planetary Ephemerides on an Electronic Computer

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LINCOLN LABORATORY

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M. E. ASH

Group 63

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ABSTRACT

A computer program, called the Planetary Ephemeris Program (PEP), is being written at Lincoln Laboratory. The purpose of the program is to improve planetary and lunar ephemerides using the results of radar and optical observations. In this report, we derive the differential equations that are numerically integrated in PEP to determine as functions of time the positions and velocities of the planets, of the Earth-Moon barycenter and of the Moon, and the partial derivatives of these positions and velocities with respect to initial conditions, masses and other parameters. Newtonian theory with the usual unrigorous general relativistic corrections is employed. The equations of motion and the equations for the partial derivatives with respect to initial conditions are presented in the form needed in the Encke's method of integration used in PEP.

Accepted for the Air Force
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TABLE OF CONTENTS

	Abstract	iii
I.	INTRODUCTION	1
II.	ELLIPTIC MOTION	5
	A. Change of Coordinates	5
	B. Determination of Position and Velocity at a Given Time from Orbital Elements	6
	C. Determination of Orbital Elements from Position and Velocity at a Given Time	7
	D. Partial Derivatives of Position and Velocity with Respect to Orbital Elements	9
	E. Partial Derivatives of Orbital Elements with Respect to Position and Velocity	13
	F. Check of Elliptic Orbit Formulas	16
III.	NEWTONIAN EQUATIONS OF MOTION	19
	A. Two-Body Problem (Planet-Sun) with Perturbing Forces	19
	B. Three-Body Problem (Earth-Moon-Sun) with Perturbing Forces	20
	C. Planetary Perturbations	22
	D. Second Harmonic of the Gravitational Potential of the Sun	28
	E. Higher Harmonics in the Gravitational Potentials of the Earth and Moon	30
IV.	GENERAL RELATIVISTIC EFFECT	39
	A. Mathematical Formalism of Einstein's General Theory of Relativity	39
	B. Motion of a Planet of Small Mass in the Gravitational Field of the Sun	43
	C. Method of Solving the Problem of the Motion of a System of Masses in General Relativity	50
V.	EQUATIONS FOR PARTIAL DERIVATIVES OF POSITION AND VELOCITY WITH RESPECT TO INITIAL CONDITIONS AND PARAMETERS	53
	A. Planet Case	53
	B. Earth-Moon Case	56
VI.	ENCKE'S EQUATIONS OF MOTION AND ENCKE'S EQUATIONS FOR PARTIAL DERIVATIVES WITH RESPECT TO INITIAL CONDITIONS	67
	A. Planet Case	67
	B. Earth-Moon Case	69
APPENDIX A –	Precession-Nutation of the Earth	73
APPENDIX B –	Rotation and Physical Libration of the Moon	75
APPENDIX C –	Orientation of the Sun	79

GENERATION OF PLANETARY EPHEMERIDES ON AN ELECTRONIC COMPUTER

I. INTRODUCTION

A computer program, called the Planetary Ephemeris Program (PEP), is being written at Lincoln Laboratory. The purpose of the program is to improve planetary and lunar ephemerides using the results of radar and optical observations. The procedure for improving ephemerides is as follows. First, we integrate the differential equations of motion of the planets, of the Earth-Moon barycenter and of the Moon using provisional values for the various parameters (such as initial conditions and masses) appearing in the theory of gravitation and motion employed. We also integrate the differential equations for the partial derivatives of the position and velocity of the bodies with respect to these parameters. Then, for each radar and optical observation, we calculate the theoretical values of the measurements made in the observation and the partial derivatives of these theoretical values with respect to the parameters. Using the differences between the observed and theoretical values of the measurements, the errors of the measurements and the partial derivatives of the theoretical values of the measurements, we form the normal equations and solve them to get corrections to the parameters. With the adjusted parameters, we reintegrate the equations of motion and the equations for the partial derivatives; applying the results of these integrations, we again form the normal equations and solve them to get further corrections to the parameters. This process is repeated until we obtain convergence. Using the parameters thus obtained in the integration of the equations of motion, we generate ephemerides which best agree with the observations in a least-squares sense.

In this report, we present derivations of the formulas used in PEP to determine as functions of time the positions and velocities of the planets, of the Earth-Moon barycenter and of the Moon, and the partial derivatives of these positions and velocities with respect to the various parameters. In a second report, we will describe the derivations of the formulas used in the comparison of theory and observation and in the least-squares analysis parts of the program. In a third report, we will present the documentation of the computer program itself.

In deriving the equations in this report, we employ Newton's theory of gravitation and motion with the usual unrigorous general relativistic corrections. Since one of our purposes in analyzing radar and optical observations with PEP is to test general relativity, the equations of motion employed should be derived in strict accordance with the principles of this theory. As explained in Sec. IV-C, the equations of motion of the general relativistic N-body problem have been derived in principle; it is only necessary to learn the derivation and to put the equations in a form amenable to the generation of ephemerides. The equations will resemble the Newtonian equations with rigorous rather than unrigorous general relativistic corrections. The rigorous general relativistic treatment will be presented in a subsequent report.

In PEP, the position and velocity of a planet, of the Earth-Moon barycenter or of the Moon are determined as functions of time by numerically integrating the Encke differential equations for these quantities, with the positions of perturbing planets being determined during the integration from an input magnetic tape. The partial derivatives of position and velocity with respect to masses and other parameters (not initial conditions) are determined as functions of time by numerically integrating the differential equations for these quantities, while those with respect to initial conditions are determined as functions of time either by assuming that they are equal to the partial derivatives of position and velocity with respect to initial conditions in the elliptic orbit osculating to the true orbit of the body at the initial time, or by numerically integrating the Encke equations for these quantities.

We feel that numerical integration on an electronic computer using Encke's method can yield centuries of planetary and Earth-Moon barycenter ephemerides of the accuracy required by the observations to which the ephemerides must be fitted. In the case of the Moon, however, numerical integration of Encke's equations might not yield an ephemeris accurate for centuries, although it certainly would have the required accuracy for decades. Thus, we would have to manipulate the equations further into a form that would yield accurate results for centuries of numerical integration.†

We have not used the traditional method of obtaining planetary motions by expansions in series for a number of reasons. First, the higher accuracies we require necessitate a higher-order perturbation theory and many more terms in the truncated series than were required when the present ephemerides were generated. Further, the equations we numerically integrate are directly derivable from the theory of gravitation and motion employed, while many operations have to be carried out to derive the series solutions, thus introducing the possibility of error. Finally, it is easy to introduce additional forces in the numerically integrated equations, whereas the consideration of the effect of additional forces on the series solutions requires much effort.

PEP could be expanded to numerically integrate the equations for the motion of the Earth about its center of mass, in addition to the equations for the motions of planets around the Sun, of the Earth-Moon barycenter around the Sun and of the Moon around the Earth. In this way, all the present ephemerides of the motions in the solar system could be completely and rigorously revised, instead of only revising the ephemerides of the motions of the center of masses of the various bodies, assuming the present expressions for the rotation and precession-nutation of the Earth. However, even if we assume these expressions, significant improvements in the ephemerides of the motions of the center of masses can be made.

Recent radar observations at Lincoln Laboratory and elsewhere of Mercury, Venus, Mars and the Moon‡ have much greater accuracy than optical observations of these bodies. However, optical observations have the advantage of having been made over a period of several centuries. Using both radar and optical observations to improve ephemerides takes account of the stated advantages of both kinds of observations. In addition, the dimensionality of the space of observations is increased to four over the two obtainable using only optical observations; that is, radar observations of time delay and doppler shift give range and range-rate measurements in addition to the two angular measurements given by optical observations.

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† These manipulations have since been performed and will be presented in: M. E. Ash, "Generation of the Lunar Ephemeris on an Electronic Computer," Technical Report 400, Lincoln Laboratory, M.I.T. (24 August 1965).

‡ The Sun and perhaps Jupiter have also been detected with radar, but the results of such observations are not yet of the accuracy or nature needed in improving ephemerides.

With PEP, the fact that fitting ephemerides to observational data is done by an electronic computer and is completely automated implies that more accurate ephemerides will be generated than if traditional methods (largely dependent upon hand computation) were used, even with exactly the same observational data as input. Of course, more observational data are available now than when the present ephemerides were generated, even without counting radar observations.

In the process of improving ephemerides, we obtain improved values of the various parameters appearing in the theory (such as planetary and lunar masses); we also test the validity of the theory employed. Some hypotheses which we are interested in testing are:

(1) Does the Sun have a detectable second harmonic in its gravitational potential?

(2) Are the values of the gravitational constant and the velocity of light functions of time?

(3) Is there an advance of the perihelion of the orbit of Mercury and the other planets as predicted by general relativity?

(4) Is the general relativistic expression for the time delay of a radar signal passing near the Sun correct?[1]

(5) Can atomic time be identified with the proper time of general relativity?

The prediction of the advance of the perihelion of Mercury's orbit has been verified previously. Since experimental results are supposed to be reproducible, and since we intend to check this advance with more accurate data, the effort we make to do this is not without merit. In order to test the general relativistic effect on the time delay of the radar signal, we need observations of Venus or Mercury at superior conjunction at the frequency of Lincoln's Haystack radar (8×10^9 cps). These have not yet been made, but hopefully will be made in the next few years.

II. ELLIPTIC MOTION

A. CHANGE OF COORDINATES

In Fig. 1, Ω is the longitude of the ascending node of an elliptic orbit, i is the angle of inclination of the orbital plane, and ω is the argument of perigee. We wish to find the transformation from the (x, y, z) coordinate system to the $(\bar{x}, \bar{y}, \bar{z})$ coordinate system, whose $\bar{x}$-axis is pointed in the direction of perigee, whose $\bar{y}$-axis is pointed in the direction of motion at perigee, and whose $\bar{z}$-axis is perpendicular to the orbital plane.

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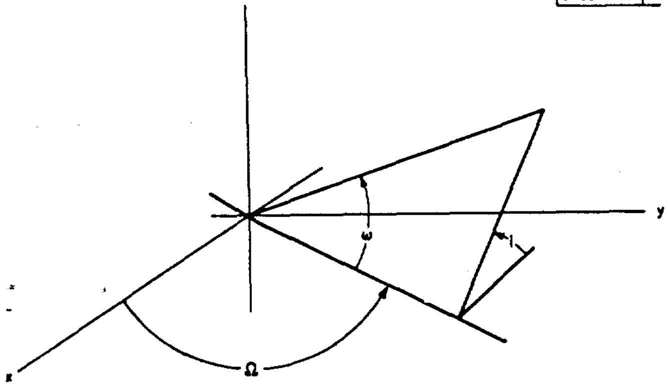

Fig. 1. Euler angles.

If we rotate the xy-plane about the z-axis through the angle Ω, we obtain an (x', y', z') coordinate system related to the original one by the equations

$$x' = x \cos\Omega + y \sin\Omega$$
$$y' = -x \sin\Omega + y \cos\Omega$$
$$z' = z \quad .$$

Rotating about the x'-axis through the angle i, we obtain

$$x'' = x'$$
$$y'' = y' \cos i + z' \sin i$$
$$z'' = -y' \sin i + z' \cos i \quad .$$

Finally, rotating about the z''-axis through the angle ω, we see that

$$\bar{x} = x'' \cos\omega + y'' \sin\omega$$
$$\bar{y} = -x'' \sin\omega + y'' \cos\omega$$
$$\bar{z} = z'' \quad .$$

The net result of these three transformations is seen to be

$$\bar{x} = (\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i)\, x + (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i)\, y + (\sin\omega \sin i)\, z$$
$$\bar{y} = -(\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i)\, x + (-\sin\Omega \sin\omega + \cos\Omega \cos\omega \cos i)\, y + (\cos\omega \sin i)\, z$$
$$\bar{z} = (\sin\Omega \sin i)\, x - (\cos\Omega \sin i)\, y + (\cos i)\, z \quad . \qquad \text{(II-1)}$$

Equation (II-1) is an orthogonal transformation, so its inverse transformation is given by the matrix which is the transpose of the above matrix. Thus, we have

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$$x = (\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i)\,\bar{x} - (\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i)\,\bar{y} + (\sin\Omega \sin i)\,\bar{z}$$

$$y = (\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i)\,\bar{x} + (-\sin\Omega \sin\omega + \cos\Omega \cos\omega \cos i)\,\bar{y} - (\cos\Omega \sin i)\,\bar{z}$$

$$z = (\sin\omega \sin i)\,\bar{x} + (\cos\omega \sin i)\,\bar{y} + (\cos i)\,\bar{z} \quad . \tag{II-2}$$

These formulas agree with those given in Ref. 2.

B. DETERMINATION OF POSITION AND VELOCITY AT A GIVEN TIME FROM ORBITAL ELEMENTS

The differential equation system

$$\frac{d^2y^2}{dt^2} = -\frac{\mu y^j}{\rho^3} \quad , \quad j = 1, 2, 3 \tag{II-3}$$

where $\mu > 0$ and $\rho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$, represents the motion of a body traveling in a conic section. We shall assume that this conic section is an ellipse.

The various parameters used to describe elliptic motion are:

a = semimajor axis

e = eccentricity ($0 \leqslant e < 1$)

i = orbital inclination ($0 \leqslant i \leqslant 180°$)

Ω = longitude of the ascending node ($0 \leqslant \Omega < 360°$)

ω = argument of perigee, measured along the orbital plane from the ascending node ($0° \leqslant \omega < 360°$). (Note: the longitude of perigee is the quantity $\tilde{\omega} = \Omega + \omega$.)

ℓ_o = initial mean anomaly at time t_o ($0° \leqslant \ell_o < 360°$)

ℓ = mean anomaly at time t ($0° \leqslant \ell < 360°$)

f = true anomaly at time t ($0° \leqslant f < 360°$)

u = eccentric anomaly at time t ($0° \leqslant u < 360°$)

n = mean motion

p = semilatus rectum

The quantities $(a, e, i, \Omega, \omega, \ell_o)$ are the elements of the elliptic orbit and, along with the time, completely determine the position and velocity $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ in the orbit.

Let $(\bar{x}, \bar{y})$ be the Cartesian coordinate system in the orbital plane whose $\bar{x}$-axis points in the direction of perigee and whose $\bar{y}$-axis points in the direction of motion at perigee. The following equations are derived in Ref. 3 (the formula numbers in square brackets are those in the reference).

$$(\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2 = \mu\left(\frac{2}{\rho} - \frac{1}{a}\right) \quad [26, 41] \tag{II-4}$$

$$p = a(1 - e^2) \tag{II-5}$$

$$\rho = \frac{p}{1 + e\cos f} \quad [44] \tag{II-6}$$

$$\rho = a(1 - e\cos u) \quad [51] \tag{II-7}$$

$$n = \mu^{1/2} a^{-3/2} \quad [43] \tag{II-8}$$

$$\ell = \ell_o + n(t - t_o) \quad [50] \tag{II-9}$$

$$\ell = u - e \sin u \qquad [49] \qquad \text{(II-10)}$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \qquad [52] \qquad \text{(II-11)}$$

$$\bar{x} = a(\cos u - e) \qquad [54] \qquad \text{(II-12)}$$

$$\bar{y} = a\sqrt{1-e^2}\sin u \qquad [55] \quad . \qquad \text{(II-13)}$$

These are the basic formulas which we shall assume to be known. Differentiating (II-9) and (II-10) with respect to time, we find that

$$n = \frac{d\ell}{dt} = (1 - e\cos u)\frac{du}{dt} \quad .$$

Thus,

$$\frac{du}{dt} = \frac{na}{\rho} \quad . \qquad \text{(II-14)}$$

Equations (II-12), (II-13) and (II-14) together give

$$\dot{\bar{x}} = -\frac{na^2}{\rho}\sin u \qquad \text{(II-15)}$$

$$\dot{\bar{y}} = \frac{na^2\sqrt{1-e^2}}{\rho}\cos u \quad . \qquad \text{(II-16)}$$

Suppose we are given $(a, e, i, \Omega, \omega, \ell_o)$, and we wish to find $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at time t. Equation (II-9) determines the mean anomaly ℓ at time t, from which Kepler's equation (II-10) (solved by iteration) gives the eccentric anomaly u at time t. Then, formulas (II-7), (II-12), (II-13), (II-15) and (II-16) determine $\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}$ at time t. If we define

$$\begin{aligned}
b_{11} &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\
b_{12} &= -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\
b_{21} &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\
b_{22} &= -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\
b_{31} &= \sin\omega\sin i \\
b_{32} &= \cos\omega\sin i
\end{aligned} \qquad \text{(II-17)}$$

we finally obtain, by (II-2),

$$\left.\begin{aligned}
y^j &= b_{j1}\bar{x} + b_{j2}\bar{y} \\
\dot{y}^j &= b_{j1}\dot{\bar{x}} + b_{j2}\dot{\bar{y}}
\end{aligned}\right\} \quad j = 1, 2, 3 \quad . \qquad \text{(II-18)}$$

C. DETERMINATION OF ORBITAL ELEMENTS FROM POSITION AND VELOCITY AT A GIVEN TIME

Suppose we are given $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at time t, and are required to find $(a, e, i, \Omega, \omega, \ell_o)$, with ℓ_o being the mean anomaly at time t_o.

We define

$$\rho^2 = (y^1)^2 + (y^2)^2 + (y^3)^2$$

$$v^2 = (\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2$$

$$\vec{\rho} \cdot \vec{v} = y^1\dot{y}^1 + y^2\dot{y}^2 + y^3\dot{y}^3 \quad . \tag{II-19}$$

Then the vis viva integral (II-4) immediately determines one of the elements:

$$a = \frac{\mu}{\frac{2\mu}{\rho} - v^2} \quad . \tag{II-20}$$

Differentiating (II-7) with respect to time, we obtain

$$\frac{d\rho}{dt} = ae \sin u \, \frac{du}{dt} = \frac{na^2 e \sin u}{\rho}$$

by (II-14). Since

$$\frac{d\rho}{dt} = \frac{\vec{\rho} \cdot \vec{v}}{\rho} \quad ,$$

this gives

$$e \sin u = \frac{\vec{\rho} \cdot \vec{v}}{na^2} \quad . \tag{II-21}$$

Further, (II-7) can be put in the form

$$e \cos u = 1 - \frac{\rho}{a} \quad . \tag{II-22}$$

The simultaneous solution of (II-21) and (II-22) determines e and u. The mean anomaly ℓ_o at time t_o is then found from Kepler's equation (II-10), and formulas (II-12) through (II-16) determine $\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}$. Solving (II-18) for the b_{jk}, we have

$$\left.\begin{aligned} b_{j1} &= \frac{1}{(\bar{x}\dot{\bar{y}} - \dot{\bar{x}}\bar{y})} (y^j\dot{\bar{y}} - \dot{y}^j\bar{y}) \\ b_{j2} &= \frac{1}{(\bar{x}\dot{\bar{y}} - \dot{\bar{x}}\bar{y})} (\dot{y}^j\bar{x} - y^j\dot{\bar{x}}) \end{aligned}\right\} \quad j = 1, 2, 3$$

which can be put in the form

$$\left.\begin{aligned} b_{j1} &= y^j\left(\frac{\cos u}{\rho}\right) - \dot{y}^j\left(\frac{\sin u}{na}\right) \\ b_{j2} &= y^j\left(\frac{\sin u}{\rho\sqrt{1 - e^2}}\right) + \dot{y}^j\left(\frac{\cos u - e}{na\sqrt{1 - e^2}}\right) \end{aligned}\right\} \quad j = 1, 2, 3 \quad . \tag{II-23}$$

These formulas agree with those in Ref. 4, in which $\sqrt{\mu}\, t$ was used as the time variable instead of t.

By means of the above methods we can thus determine a, e, ℓ_o, and b_{jk} (j = 1, 2, 3; k = 1, 2) given the position and velocity $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at time t. From these quantities, we can then determine the values of $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at any time from the formulas given in Sec. II-B above.

To determine i, Ω and ω, we must solve (II-17). First, we have

$$\left.\begin{aligned} \sin i &= \sqrt{b_{31}^2 + b_{32}^2} \\ \cos i &= b_{11}b_{22} - b_{12}b_{21} \end{aligned}\right\} \quad 0° \leqslant i \leqslant 180° \quad . \tag{II-24}$$

Further, if $i \neq 0°$ or $180°$, we see that ω is determined by the relations

$$\left.\begin{aligned} \sin \omega &= \frac{b_{31}}{\sqrt{b_{31}^2 + b_{32}^2}} \\ \cos \omega &= \frac{b_{32}}{\sqrt{b_{31}^2 + b_{32}^2}} \end{aligned}\right\} \quad 0° \leqslant \omega < 360° \quad . \tag{II-25}$$

Finally, we see that Ω is determined by the relations

$$\left.\begin{aligned} \sin \Omega &= b_{21} \cos \omega - b_{22} \sin \omega \\ \cos \Omega &= b_{11} \cos \omega - b_{12} \sin \omega \end{aligned}\right\} \quad 0° \leqslant \Omega < 360° \quad . \tag{II-26}$$

If $i = 0°$ or $180°$, the equations in (II-17) assume the form

$$\begin{aligned} b_{11} &= \cos(\Omega \pm \omega) & b_{22} &= \pm\cos(\Omega \pm \omega) \\ b_{12} &= -\sin(\omega \pm \Omega) & b_{31} &= 0 \\ b_{21} &= \sin(\Omega \pm \omega) & b_{32} &= 0 \end{aligned}$$

where the plus sign of the $\pm$ symbol is to be used if $i = 0°$, and the minus sign is to be used if $i = 180°$. Thus, Ω and ω are indeterminate when $i = 0°$ or $180°$. We might make the convention that when $i = 0°$ or $180°$, we set $\Omega = 0$ and determine ω by the relations

$$\begin{aligned} \cos \omega &= b_{11} \\ \sin \omega &= -b_{12} \quad . \end{aligned} \tag{II-27}$$

D. PARTIAL DERIVATIVES OF POSITION AND VELOCITY WITH RESPECT TO ORBITAL ELEMENTS

Regarding the position and velocity $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at time t as functions of the orbital elements $(a, e, i, \Omega, \omega, \ell_o)$, we derive the following equations.

$$\left.\begin{aligned} \frac{\partial y^j}{\partial a} &= \frac{y^j}{a} - \frac{3}{2}\frac{\dot{y}^j}{a}(t - t_o) \\ \frac{\partial \dot{y}^j}{\partial a} &= -\dot{y}^j\left[\frac{1}{2a} + \frac{3e\dot{\overline{x}}(t - t_o)}{2a\rho}\right] + \frac{3n(t - t_o)}{2\rho}\left(b_{j1}\frac{\dot{\overline{y}}}{\sqrt{1 - e^2}} - b_{j2}\sqrt{1 - e^2}\,\dot{\overline{x}}\right) \end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-28}$$

$$\left.\begin{aligned} \frac{\partial y^j}{\partial e} &= -ab_{j1} - \frac{e}{1 - e^2} b_{j2}\,\overline{y} + \frac{\dot{y}^j \sin u}{n} \\ \frac{\partial \dot{y}^j}{\partial e} &= \frac{\dot{y}^j}{\rho}\left(a \cos u + \frac{e\dot{\overline{x}} \sin u}{n}\right) + \frac{b_{j1}a\dot{\overline{x}} \cos u}{\rho} + b_{j2}\left(\frac{\dot{\overline{x}}\dot{\overline{y}}}{\rho} - \frac{e\dot{\overline{y}}}{1 - e^2}\right) \end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-29}$$

$$\left.\begin{aligned}\frac{\partial y^j}{\partial \ell_o} &= \frac{\dot{y}^j}{n} \\ \frac{\partial \dot{y}^j}{\partial \ell_o} &= \frac{\dot{y}^j e \dot{\bar{x}}}{n\rho} + \frac{a}{\rho}\left(-b_{j1}\frac{\dot{\bar{y}}}{\sqrt{1-e^2}} + b_{j2}\sqrt{1-e^2}\,\dot{\bar{x}}\right)\end{aligned}\right\}\quad j = 1, 2, 3 \tag{II-30}$$

$$\left.\begin{aligned}\frac{\partial y^1}{\partial i} &= (\sin\Omega)\, y^3 & \frac{\partial \dot{y}^1}{\partial i} &= (\sin\Omega)\, \dot{y}^3 \\ \frac{\partial y^2}{\partial i} &= -(\cos\Omega)\, y^3 & \frac{\partial \dot{y}^2}{\partial i} &= -(\cos\Omega)\, \dot{y}^3 \\ \frac{\partial y^3}{\partial i} &= (\sin\omega\ \cos i)\, \bar{x} & \frac{\partial \dot{y}^3}{\partial i} &= (\sin\omega\ \cos i)\, \dot{\bar{x}} \\ &\quad + (\cos\omega\ \cos i)\, \bar{y} & &\quad + (\cos\omega\ \cos i)\, \dot{\bar{y}}\end{aligned}\right\} \tag{II-31}$$

$$\left.\begin{aligned}\frac{\partial y^1}{\partial \Omega} &= -y^2 & \frac{\partial \dot{y}^1}{\partial \Omega} &= -\dot{y}^2 \\ \frac{\partial y^2}{\partial \Omega} &= y^1 & \frac{\partial \dot{y}^2}{\partial \Omega} &= \dot{y}^1 \\ \frac{\partial y^3}{\partial \Omega} &= 0 & \frac{\partial \dot{y}^3}{\partial \Omega} &= 0\end{aligned}\right\} \tag{II-32}$$

$$\left.\begin{aligned}\frac{\partial y^j}{\partial \omega} &= b_{j2}\bar{x} - b_{j1}\bar{y} \\ \frac{\partial \dot{y}^j}{\partial \omega} &= b_{j2}\dot{\bar{x}} - b_{j1}\dot{\bar{y}}\end{aligned}\right\}\quad j = 1, 2, 3 \quad . \tag{II-33}$$

To show the validity of the above formulas, we note that in (II-18) the b_{jk} are functions of i, Ω, ω, and $\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}$ are functions of a, e, ℓ_o, t. Therefore, we first differentiate (II-17) with respect to i, Ω, ω, and find that

$$\begin{aligned}\frac{\partial b_{11}}{\partial i} &= \sin\Omega\ \sin\omega\ \sin i = (\sin\Omega)\ b_{31} \\ \frac{\partial b_{12}}{\partial i} &= \sin\Omega\ \cos\omega\ \sin i = (\sin\Omega)\ b_{32} \\ \frac{\partial b_{21}}{\partial i} &= -\cos\Omega\ \sin\omega\ \sin i = -(\cos\Omega)\ b_{31} \\ \frac{\partial b_{22}}{\partial i} &= -\cos\Omega\ \cos\omega\ \sin i = -(\cos\Omega)\ b_{32} \\ \frac{\partial b_{31}}{\partial i} &= \sin\omega\ \cos i \\ \frac{\partial b_{32}}{\partial i} &= \cos\omega\ \cos i \quad ;\end{aligned} \tag{II-34}$$

$$\frac{\partial b_{11}}{\partial \Omega} = -\sin\Omega \cos\omega - \cos\Omega \sin\omega \cos i = -b_{21}$$

$$\frac{\partial b_{12}}{\partial \Omega} = \sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i = -b_{22}$$

$$\frac{\partial b_{21}}{\partial \Omega} = \cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i = b_{11}$$

$$\frac{\partial b_{22}}{\partial \Omega} = -\cos\Omega \sin\omega - \sin\Omega \cos\omega \cos i = b_{12}$$

$$\frac{\partial b_{31}}{\partial \Omega} = 0$$

$$\frac{\partial b_{32}}{\partial \Omega} = 0 \quad ; \tag{II-35}$$

$$\frac{\partial b_{11}}{\partial \omega} = -\cos\Omega \sin\omega - \sin\Omega \cos\omega \cos i = b_{12}$$

$$\frac{\partial b_{12}}{\partial \omega} = -\cos\Omega \cos\omega + \sin\Omega \sin\omega \cos i = -b_{11}$$

$$\frac{\partial b_{21}}{\partial \omega} = -\sin\Omega \sin\omega + \cos\Omega \cos\omega \cos i = b_{22}$$

$$\frac{\partial b_{22}}{\partial \omega} = -\sin\Omega \cos\omega - \cos\Omega \sin\omega \cos i = -b_{21}$$

$$\frac{\partial b_{31}}{\partial \omega} = \cos\omega \sin i = b_{32}$$

$$\frac{\partial b_{32}}{\partial \omega} = -\sin\omega \sin i = -b_{31} \quad . \tag{II-36}$$

Then, (II-34) through (II-36) and (II-18) together show that formulas (II-31) through (II-33) are valid.

Next, by (II-8), we have

$$\frac{\partial n}{\partial a} = -\frac{3}{2}\frac{n}{a} \quad . \tag{II-37}$$

Further, (II-9) and (II-10) give

$$-\frac{3}{2}\frac{n}{a}(t - t_o) = \frac{\partial \ell}{\partial a} = \frac{\partial u}{\partial a} - e \cos u \frac{\partial u}{\partial a}$$

$$0 = \frac{\partial \ell}{\partial e} = \frac{\partial u}{\partial e} - \sin u - e \cos u \frac{\partial u}{\partial e}$$

$$1 = \frac{\partial \ell}{\partial \ell_o} = \frac{\partial u}{\partial \ell_o} - e \cos u \frac{\partial u}{\partial \ell_o} \quad .$$

By (II-7), these equations imply that

$$\frac{\partial u}{\partial a} = -\frac{3}{2}\,\frac{n(t-t_o)}{\rho}$$

$$\frac{\partial u}{\partial e} = \frac{a \sin u}{\rho}$$

$$\frac{\partial u}{\partial \ell_o} = \frac{a}{\rho} \quad . \tag{II-38}$$

From (II-7) and (II-38), it follows that

$$\frac{\partial \rho}{\partial a} = (1 - e \cos u) + ae \sin u \,\frac{\partial u}{\partial a} = \frac{\rho}{a} - \frac{3}{2}\,\frac{nae(t-t_o)\sin u}{\rho}$$

$$\frac{\partial \rho}{\partial e} = -a \cos u + ae \sin u \,\frac{\partial u}{\partial e} = -a \cos u + \frac{a^2 e \sin^2 u}{\rho}$$

$$\frac{\partial \rho}{\partial \ell_o} = ae \sin u \,\frac{\partial u}{\partial \ell_o} = \frac{a^2 e \sin u}{\rho} \quad . \tag{II-39}$$

Having obtained the above formulas, we use (II-12) to (II-16) to find the derivatives of $\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}$ with respect to a, e, ℓ_o.

$$\frac{\partial \bar{x}}{\partial a} = (\cos u - e) - a \sin u \,\frac{\partial u}{\partial a} = \frac{\bar{x}}{a} - \frac{3}{2}\,\frac{\dot{\bar{x}}}{a}(t-t_o)$$

$$\frac{\partial \bar{x}}{\partial e} = -a - a \sin u \,\frac{\partial u}{\partial e} = -a + \frac{\dot{\bar{x}} \sin u}{n}$$

$$\frac{\partial \bar{x}}{\partial \ell_o} = -a \sin u \,\frac{\partial u}{\partial \ell_o} = \frac{\dot{\bar{x}}}{n} \quad ; \tag{II-40}$$

$$\frac{\partial \bar{y}}{\partial a} = \sqrt{1-e^2}\,\sin u + a\sqrt{1-e^2}\,\cos u \,\frac{\partial u}{\partial a} = \frac{\bar{y}}{a} - \frac{3}{2}\,\frac{\dot{\bar{y}}}{a}(t-t_o)$$

$$\frac{\partial \bar{y}}{\partial e} = -\frac{ae \sin u}{\sqrt{1-e^2}} + a\sqrt{1-e^2}\,\cos u \,\frac{\partial u}{\partial e} = -\frac{e\bar{y}}{1-e^2} + \frac{\dot{\bar{y}} \sin u}{n}$$

$$\frac{\partial \bar{y}}{\partial \ell_o} = a\sqrt{1-e^2}\,\cos u \,\frac{\partial u}{\partial \ell_o} = \frac{\dot{\bar{y}}}{n} \quad ; \tag{II-41}$$

$$\begin{aligned}
\frac{\partial \dot{\bar{x}}}{\partial a} &= -\frac{a^2 \sin u}{\rho}\,\frac{\partial n}{\partial a} - \frac{2na \sin u}{\rho} - \frac{na^2 \cos u}{\rho}\,\frac{\partial u}{\partial a} + \frac{na^2 \sin u}{\rho^2}\,\frac{\partial \rho}{\partial a} \\
&= \dot{\bar{x}}\left(-\frac{3}{2a} + \frac{2}{a}\right) + \frac{3n^2a^2 \cos u\,(t-t_o)}{2\rho^2} - \dot{\bar{x}}\left[\frac{1}{a} - \frac{3nae \sin u\,(t-t_o)}{2\rho^2}\right] \\
&= -\dot{\bar{x}}\left[\frac{1}{2a} + \frac{3e\dot{\bar{x}}(t-t_o)}{2a\rho}\right] + \frac{3n\dot{\bar{y}}(t-t_o)}{2\rho\sqrt{1-e^2}} \quad ;
\end{aligned} \tag{II-42}$$

$$\frac{\partial \dot{x}}{\partial e} = \frac{-na^2 \cos u}{\rho} \frac{\partial u}{\partial e} + \frac{na^2 \sin u}{\rho^2} \frac{\partial \rho}{\partial e} = \frac{a\dot{x}\cos u}{\rho} + \frac{\dot{x}}{\rho}\left(a \cos u + \frac{e\dot{x}\sin u}{n}\right)$$

$$\frac{\partial \dot{x}}{\partial \ell_o} = -\frac{na^2 \cos u}{\rho} \frac{\partial u}{\partial \ell_o} + \frac{na^2 \sin u}{\rho^2} \frac{\partial \rho}{\partial \ell_o} = -\frac{a\dot{y}}{\rho\sqrt{1-e^2}} + \frac{\dot{x}e\dot{x}}{n\rho} \quad ; \tag{II-43}$$

$$\frac{\partial \dot{y}}{\partial a} = \frac{a^2\sqrt{1-e^2}\cos u}{\rho}\frac{\partial n}{\partial a} + \frac{2na\sqrt{1-e^2}\cos u}{\rho} - \frac{na^2\sqrt{1-e^2}\sin u}{\rho}\frac{\partial u}{\partial a}$$

$$- \frac{na^2\sqrt{1-e^2}\cos u}{\rho^2}\frac{\partial \rho}{\partial a}$$

$$= \dot{y}\left(-\frac{3}{2a} + \frac{2}{a}\right) + \frac{3n^2a^2\sqrt{1-e^2}\sin u(t-t_o)}{2\rho^2} - \dot{y}\left[\frac{1}{a} - \frac{3nae\sin u(t-t_o)}{2\rho^2}\right]$$

$$= -\dot{y}\left[\frac{1}{2a} + \frac{3e\dot{x}(t-t_o)}{2a\rho}\right] - \frac{3n\sqrt{1-e^2}\dot{x}(t-t_o)}{2\rho} \quad ; \tag{II-44}$$

$$\frac{\partial \dot{y}}{\partial e} = -\frac{na^2 e\cos u}{\rho\sqrt{1-e^2}} - \frac{na^2\sqrt{1-e^2}\sin u}{\rho}\frac{\partial u}{\partial e} - \frac{na^2\sqrt{1-e^2}\cos u}{\rho^2}\cdot\frac{\partial \rho}{\partial e}$$

$$= -\frac{e\dot{y}}{(1-e^2)} + \frac{\dot{x}\dot{y}}{\rho} + \frac{\dot{y}}{\rho}\left(a\cos u + \frac{e\dot{x}\sin u}{n}\right)$$

$$\frac{\partial \dot{y}}{\partial \ell_o} = -\frac{na^2\sqrt{1-e^2}\cos u}{\rho^2}\frac{\partial \rho}{\partial \ell_o} - \frac{na^2\sqrt{1-e^2}\sin u}{\rho}\frac{\partial u}{\partial \ell_o} = \frac{\dot{y}e\dot{x}}{n\rho} + \frac{a\sqrt{1-e^2}\dot{x}}{\rho} \quad . \tag{II-45}$$

Equations (II-40) through (II-45) and (II-18) together show that formulas (II-28) through (II-30) are valid.

E. PARTIAL DERIVATIVES OF ORBITAL ELEMENTS WITH RESPECT TO POSITION AND VELOCITY

Regarding the orbital elements $(a, e, i, \Omega, \omega, \ell_o)$ at time t_o as functions of the position and velocity $(y^1, y^2, y^3, \dot{y}^1, \dot{y}^2, \dot{y}^3)$ at time t, we derive the following equations.

$$\left.\begin{aligned} \frac{\partial a}{\partial y^j} &= \frac{2a^2 y^j}{\rho^3} \\ \frac{\partial a}{\partial \dot{y}^j} &= \frac{2a^2\dot{y}^j}{\mu} \end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-46}$$

$$\left.\begin{aligned} \frac{\partial e}{\partial y^j} &= \frac{\sin u}{na}\left[\frac{\dot{y}^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\,y^j}{\rho^3}\right] + \frac{y^j\cos u}{\rho}\left(\frac{2}{\rho} - \frac{1}{a}\right) \\ \frac{\partial e}{\partial \dot{y}^j} &= \frac{\sin u}{na}\left[\frac{y^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\,\dot{y}^j}{\mu}\right] + \frac{2\rho\dot{y}^j\cos u}{\mu} \end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-47}$$

$$\left.\begin{aligned}\frac{\partial \ell_o}{\partial y^j} &= \frac{\cos u - e}{nae}\left[\frac{\dot{y}^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\,y^j}{\rho^3}\right] - \frac{y^j \sin u}{e\rho}\left(\frac{2}{\rho} - \frac{1}{a}\right) + \frac{3nay^j(t - t_o)}{\rho^3}\\ \frac{\partial \ell_o}{\partial \dot{y}^j} &= \frac{\cos u - e}{nae}\left[\frac{y^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\,\dot{y}^j}{\mu}\right] - \frac{2\rho\dot{y}^j \sin u}{e\mu} + \frac{3na\dot{y}^j(t - t_o)}{\mu}\end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-48}$$

$$\begin{aligned}\frac{\partial i}{\partial y^j} = &\left(\frac{\partial b_{31}}{\partial y^j}\sin\omega + \frac{\partial b_{32}}{\partial y^j}\cos\omega\right)\cos i + \left(b_{12}\frac{\partial b_{21}}{\partial y^j} + b_{21}\frac{\partial b_{12}}{\partial y^j}\right.\\ &\left. - b_{11}\frac{\partial b_{22}}{\partial y^j} - b_{22}\frac{\partial b_{11}}{\partial y^j}\right)\sin i \quad , \quad j = 1, \ldots, 6\end{aligned} \tag{II-49}$$

where $(y^1, \ldots, y^6) = (y^1, \ldots, \dot{y}^3)$, and where the $\partial b_{k\ell}/\partial y^j$ are given in (II-58) and (II-59) below. The same remarks apply to the following two sets of equations.

$$\frac{\partial\omega}{\partial y^j} = \frac{1}{\sin i}\left(\frac{\partial b_{31}}{\partial y^j}\cos\omega - \frac{\partial b_{32}}{\partial y^j}\sin\omega\right) \quad , \quad j = 1, \ldots, 6 \tag{II-50}$$

$$\begin{aligned}\frac{\partial\Omega}{\partial y^j} = &\left(\frac{\partial b_{21}}{\partial y^j}\cos\omega - \frac{\partial b_{22}}{\partial y^j}\sin\omega\right)\cos\Omega - \left(\frac{\partial b_{11}}{\partial y^j}\cos\omega\right.\\ &\left. - \frac{\partial b_{12}}{\partial y^j}\sin\omega\right)\sin\Omega - \frac{\partial\omega}{\partial\alpha}\cos i \quad , \quad j = 1, \ldots, 6 \quad .\end{aligned} \tag{II-51}$$

Equations (II-46) follow directly from formula (II-20). To derive (II-47) and (II-48), we first differentiate (II-21) and (II-22), obtaining

$$\left.\begin{aligned}\frac{\partial(e\sin u)}{\partial y^j} &= \frac{\dot{y}^j}{na^2} - \frac{(\vec{\rho}\cdot\vec{v})\,y^j}{na\rho^3}\\ \frac{\partial(e\sin u)}{\partial \dot{y}^j} &= \frac{y^j}{na^2} - \frac{(\vec{\rho}\cdot\vec{v})\,\dot{y}^j}{na\mu}\end{aligned}\right\} \quad j = 1, 2, 3 \tag{II-52}$$

$$\left.\begin{aligned}\frac{\partial(e\cos u)}{\partial y^j} &= \frac{2y^j}{\rho^2} - \frac{y^j}{a\rho}\\ \frac{\partial(e\cos u)}{\partial \dot{y}^j} &= \frac{2\rho\dot{y}^j}{\mu}\end{aligned}\right\} \quad j = 1, 2, 3 \quad . \tag{II-53}$$

For any variable α, we have

$$\frac{\partial(e\sin u)}{\partial\alpha} = \sin u\,\frac{\partial e}{\partial\alpha} + e\cos u\,\frac{\partial u}{\partial\alpha}$$

$$\frac{\partial(e\cos u)}{\partial\alpha} = \cos u\,\frac{\partial e}{\partial\alpha} - e\sin u\,\frac{\partial u}{\partial\alpha}$$

which can be put in the form

$$\begin{aligned}\frac{\partial e}{\partial\alpha} &= \sin u\,\frac{\partial(e\sin u)}{\partial\alpha} + \cos u\,\frac{\partial(e\cos u)}{\partial\alpha}\\ \frac{\partial u}{\partial\alpha} &= \frac{1}{e}\left[\cos u\,\frac{\partial(e\sin u)}{\partial\alpha} - \sin u\,\frac{\partial(e\cos u)}{\partial\alpha}\right] \quad .\end{aligned} \tag{II-54}$$

Equations (II-52) through (II-54) together imply the validity of formulas (II-47) and of the following formulas.

$$\left.\begin{aligned} \frac{\partial u}{\partial y^j} &= \frac{\cos u}{nae}\left[\frac{\dot{y}^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\, y^j}{\rho^3}\right] - \frac{y^j \sin u}{e\rho}\left(\frac{2}{\rho} - \frac{1}{a}\right) \\ \frac{\partial u}{\partial \dot{y}^j} &= \frac{\cos u}{nae}\left[\frac{y^j}{a} - \frac{(\vec{\rho}\cdot\vec{v})\, \dot{y}^j}{\mu}\right] - \frac{2\rho\dot{y}^j \sin u}{e\mu} \end{aligned}\right\} \quad j = 1, 2, 3 \quad . \tag{II-55}$$

Let α be any variable. Differentiating (II-9) and (II-10) with respect to α, we see that

$$\frac{\partial \ell_o}{\partial \alpha} + \frac{\partial n}{\partial \alpha}(t - t_o) = \frac{\partial \ell}{\partial \alpha} = \frac{\partial u}{\partial \alpha} - \frac{\partial (e \sin u)}{\partial \alpha} \tag{II-56}$$

where, by (II-8),

$$\frac{\partial n}{\partial \alpha} = -\frac{3}{2}\,\frac{n}{a}\,\frac{\partial a}{\partial \alpha} \quad . \tag{II-57}$$

Formulas (II-48) then follow from (II-52) and (II-55) to (II-57).

Next, we differentiate (II-23) with respect to y^j and $\dot{y}^j$, obtaining

$$\left.\begin{aligned} \frac{\partial b_{k1}}{\partial y^j} &= \frac{\cos u}{\rho}\,\delta_{kj} - \frac{y^k y^j \cos u}{\rho^3} - \frac{\dot{y}^k \sin u}{2na^2}\,\frac{\partial a}{\partial y^j} \\ &\quad - \left(\frac{y^k \sin u}{\rho} + \frac{\dot{y}^k \cos u}{na}\right)\frac{\partial u}{\partial y^j} \\ \frac{\partial b_{k1}}{\partial \dot{y}^j} &= -\frac{\sin u}{na}\,\delta_{kj} - \frac{\dot{y}^k \sin u}{2na^2}\,\frac{\partial a}{\partial y^j} - \left(\frac{y^k \sin u}{\rho} + \frac{\dot{y}^k \cos u}{na}\right)\frac{\partial u}{\partial \dot{y}^j} \end{aligned}\right\} \quad \begin{aligned} j &= 1, 2, 3 \\ k &= 1, 2, 3 \end{aligned} \tag{II-58}$$

$$\left.\begin{aligned} \frac{\partial b_{k2}}{\partial y^j} &= \frac{1}{\sqrt{1-e^2}}\left(\frac{\sin u}{\rho}\,\delta_{kj} - \frac{y^k y^j \sin u}{\rho^3} + \frac{\dot{y}^k(\cos u - e)}{2na^2}\,\frac{\partial a}{\partial y^j}\right. \\ &\quad + \left(\frac{y^k \cos u}{\rho} - \frac{\dot{y}^k \sin u}{na}\right)\frac{\partial u}{\partial y^j} \\ &\quad \left. + \left\{\frac{y^k e \sin u}{\rho(1-e^2)} + \frac{\dot{y}^k}{na}\left[\frac{e(\cos u - e)}{(1-e^2)} - 1\right]\right\}\frac{\partial e}{\partial y^j}\right) \\ \frac{\partial b_{k2}}{\partial \dot{y}^j} &= \frac{1}{\sqrt{1-e^2}}\left(\frac{(\cos u - e)}{na}\,\delta_{kj} + \frac{\dot{y}^k(\cos u - e)}{2na^2}\,\frac{\partial a}{\partial \dot{y}^j}\right. \\ &\quad + \left(\frac{y^k \cos u}{\rho} - \frac{\dot{y}^k \sin u}{na}\right)\frac{\partial u}{\partial \dot{y}^j} \\ &\quad \left. + \left\{\frac{y^k e \sin u}{\rho(1-e^2)} + \frac{\dot{y}^k}{na}\left[\frac{e(\cos u - e)}{(1-e^2)} - 1\right]\right\}\frac{\partial e}{\partial \dot{y}^j}\right) \end{aligned}\right\} \quad \begin{aligned} j &= 1, 2, 3 \\ k &= 1, 2, 3 \end{aligned} \quad . \tag{II-59}$$

Here, the Kronecker delta δ_{kj} is defined by

$$\delta_{kj} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \quad . \tag{II-60}$$

Differentiating (II-24) with respect to y^j ($j = 1, \ldots, 6$), we see that

$$\cos i \frac{\partial i}{\partial y^j} = \frac{\partial b_{31}}{\partial y^j} \sin\omega + \frac{\partial b_{32}}{\partial y^j} \cos\omega$$

$$-\sin i \frac{\partial i}{\partial y^j} = b_{11} \frac{\partial b_{22}}{\partial y^j} + b_{22} \frac{\partial b_{11}}{\partial y^j} - b_{12} \frac{\partial b_{21}}{\partial y^j} - b_{21} \frac{\partial b_{12}}{\partial y^j} \quad . \qquad \text{(II-61)}$$

Multiplying the first equation of (II-61) by $\cos i$ and the second equation by $\sin i$, and subtracting, we obtain (II-49). Further, differentiating either the first or the second equation of (II-25) yields (II-50). Finally, by differentiating (II-26) we see that

$$\cos\Omega \frac{\partial \Omega}{\partial y^j} = \frac{\partial b_{21}}{\partial y^j} \cos\omega - \frac{\partial b_{22}}{\partial y^j} \sin\omega - (b_{21} \sin\omega + b_{22} \cos\omega) \frac{\partial \omega}{\partial y^j}$$

$$-\sin\Omega \frac{\partial \Omega}{\partial y^j} = \frac{\partial b_{11}}{\partial y^j} \cos\omega - \frac{\partial b_{12}}{\partial y^j} \sin\omega - (b_{11} \sin\omega + b_{12} \cos\omega) \frac{\partial \omega}{\partial y^j} \quad . \qquad \text{(II-62)}$$

Multiplying the first of these equations by $\cos\Omega$ and the second by $\sin\Omega$, and subtracting, we obtain (II-51). Here, we use the fact that by (II-17)

$$\cos i = (b_{21} \sin\omega + b_{22} \cos\omega) \cos\Omega - (b_{11} \sin\omega + b_{12} \cos\omega) \sin\Omega \quad .$$

F. CHECK OF ELLIPTIC ORBIT FORMULAS

The elliptic orbit formulas derived in the preceding sections were used to write computer subroutines needed in Encke's method for the numerical integration of the equations of motion and the equations for the partial derivatives with respect to initial conditions of a planet. These computer subroutines then enabled us to check the validity of the elliptic orbit formulas in the following manner.

First, note that the position and velocity $(y^1, \ldots, y^6)$ in an elliptic orbit satisfy the differential equations system

$$\left.\begin{aligned} \frac{dy^k}{dt} &= y^{k+3} \\ \frac{dy^{k+3}}{dt} &= -\frac{\mu y^k}{\rho^3} \\ y^k = y_o^k \quad , \quad y^{k+3} &= y_o^{k+3} \quad \text{when } t = t_o \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \qquad \text{(II-63)}$$

Let $(\beta^1, \ldots, \beta^6)$ denote the elements $(a, e, i, \Omega, \omega, \ell_o)$ of the elliptic orbit. Differentiating system (II-63) with respect to β^j, we obtain

$$\left.\begin{aligned} \frac{d(\partial y^k/\partial \beta^j)}{dt} &= \frac{\partial y^{k+3}}{\partial \beta^j} \\ \frac{d(\partial y^{k+3}/\partial \beta^j)}{dt} &= \frac{\mu}{\rho^3} \left(\frac{3y^k}{\rho^2} \sum_{\ell=1}^{3} y^\ell \frac{\partial y^\ell}{\partial \beta^j} - \frac{\partial y^k}{\partial \beta^j} \right) \\ \frac{\partial y^k}{\partial \beta^j} = \frac{\partial y_o^k}{\partial \beta^j} \quad , \quad \frac{\partial y^{k+3}}{\partial \beta^j} &= \frac{\partial y_o^{k+3}}{\partial \beta^j} \quad \text{when } t = t_o \end{aligned}\right\} \quad \begin{aligned} k &= 1, 2, 3 \\ j &= 1, \ldots, 6 \end{aligned} \quad . \qquad \text{(II-64)}$$

Given the orbital elements, we calculated numerically the quantities y^k and $\partial y^k/\partial\beta^j$ $(j, k = 1, \ldots, 6)$ for selected values of time using the formulas of Secs. II-B and II-D above. Also using these formulas to determine the initial conditions y_o^k and $\partial y_o^k/\partial\beta^j$ $(j, k = 1, \ldots, 6)$, we numerically integrated the 42 differential equations (II-63) and (II-64). The results of the numerical integration and the calculations from the formulas agreed to within the accuracy expected of the numerical integration, showing the validity of the formulas in Secs. II-B and II-D. This was also a check of the computer subroutines to calculate y^k and $\partial y^k/\partial\beta^j$ $(j, k = 1, \ldots, 6)$, and of the numerical integration subroutine. (We had to write our own numerical integration subroutine, since there was none available which used double precision arithmetic operations.)

Next, given the orbital elements, we calculated the position and velocity at the initial time from the formulas in Sec. II-B. Then, using the formulas of Sec. II-C, we calculated the orbital elements from this position and velocity and observed that the final and starting orbital elements agreed to within the roundoff error expected in the calculations.

Finally, given the orbital elements, we calculated the matrix $(\partial y^k/\partial\beta^j)$ from the formulas in Sec. II-D and the matrix $(\partial\beta^j/\partial y^k)$ from the formulas in Sec. II-E for selected values of time. Multiplying these two matrices, we found that the result differed from the identity matrix to within the roundoff error expected in the calculations.

All the elliptic orbit formulas derived above are valid for all choices of the orbital elements $(a, e, i, \Omega, \omega, \ell_o)$, except that some of the formulas in Sec. II-E for the matrix $(\partial\beta^j/\partial y^k)$ become indeterminant for $e = 0$ or $i = 0°$, $180°$ because here the matrix $(\partial y^k/\partial\beta^j)$ is singular. Now, the only time that the formulas for $(\partial\beta^j/\partial y^k)$ are used is when we are integrating the equations for the partial derivatives of the position and velocity of a planet with respect to initial conditions using Encke's method, and we wish to change Encke orbits. It is not very probable that, at the instant when the Encke orbit is changed, the orbital elements would be right at one of the critical points. In the case of the integration of the planetary motions, this could never happen.

III. NEWTONIAN EQUATIONS OF MOTION

A. TWO-BODY PROBLEM (PLANET-SUN) WITH PERTURBING FORCES

We wish to find the equations of motion of a celestial body about a central body, with this motion being perturbed by N other bodies (whose positions are known as functions of time) and by forces dependent on the position and velocity of the given body relative to the central body.

Let the subscript s denote the central body (s = Sun), p the given body (p = planet), and $j\,(1 \leq j \leq N)$ the j^{th} perturbing body. Let γ denote the gravitational constant, and suppose that (x^1, x^2, x^3) is an inertial coordinate system. We make the following notational conventions:

x_s^k = coordinate of central body, etc.

$x_{js}^k = x_j^k - x_s^k$ = coordinate of j relative to s, so that $x_{js}^k = -x_{sj}^k$, etc.

$r_{sj} = r_{js}$ = distance between s and j, etc.

F_p^k = k^{th} coordinate of additional force on p

M_s = mass of s, etc.

Then, by Newton's laws of motion and gravity, we have

$$\left.\begin{aligned} M_s \frac{d^2x_s^k}{dt^2} &= \gamma M_s M_p \frac{x_{ps}^k}{r_{ps}^3} + \gamma M_s \sum_{j=1}^{N} M_j \frac{x_{js}^k}{r_{js}^3} \\ M_p \frac{d^2x_p^k}{dt^2} &= \gamma M_s M_p \frac{x_{sp}^k}{r_{ps}^3} + \gamma M_p \sum_{j=1}^{N} M_j \frac{x_{jp}^k}{r_{jp}^3} + F_p^k \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{III-1}$$

By dividing the above equations by M_s and M_p, respectively, and by subtracting the first equation from the second equation, we find that the equations of motion of p relative to s are

$$\frac{d^2x_{ps}^k}{dt^2} = -\gamma M_s \left(1 + \frac{M_p}{M_s}\right) \frac{x_{ps}^k}{r_{ps}^3} + \Omega^k + \frac{1}{M_p} F_p^k \quad , \quad k = 1, 2, 3 \tag{III-2}$$

where the perturbing planet term is given by

$$\Omega^k = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{x_{jp}^k}{r_{jp}^3} - \frac{x_{js}^k}{r_{js}^3}\right) \quad , \quad k = 1, 2, 3 \quad . \tag{III-3}$$

Effects contained in the $(1/M_p)\, F_p^k$ term in (III-2) determined in this report are

R^k = general relativity effect [see (IV-52)]

S^k = second harmonic of the Sun [see (III-50)] .

If we let F_p^k denote forces acting on the planet in addition to those enumerated, (III-2) becomes

$$\frac{d^2x_{ps}^k}{dt^2} = -\gamma M_s \left(1 + \frac{M_p}{M_s}\right) \frac{x_{ps}^k}{r_{ps}^3} + \Omega^k + R^k + S^k + \frac{1}{M_p} F_p^k \quad , \quad k = 1, 2, 3 \quad . \tag{III-4}$$

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B. THREE-BODY PROBLEM (EARTH-MOON-SUN) WITH PERTURBING FORCES

We wish to find the equations of motion of the Earth-Moon barycenter about the Sun and of the Moon about the Earth, with these motions being perturbed by N planets (whose positions are known as functions of time) and by additional forces F_e and F_m acting on the Earth and Moon, respectively. We assume that these forces are expressible in terms of the relative positions and velocities of the Sun, the Earth and the Moon.

Let the subscript s denote the Sun, e the Earth, m the Moon, and j ($1 \leq j \leq N$) the j^{th} perturbing planet. Otherwise, the notation in this section is the same as in Sec. III-A above. Newton's laws of motion and gravity then give

$$\left.\begin{aligned}
\frac{d^2x_s^k}{dt^2} &= \gamma M_e \frac{x_{es}^k}{r_{es}^3} + \gamma M_m \frac{x_{ms}^k}{r_{ms}^3} + \gamma \sum_{j=1}^{N} M_j \frac{x_{js}^k}{r_{js}^3} \\
\frac{d^2x_e^k}{dt^2} &= \gamma M_s \frac{x_{se}^k}{r_{es}^3} + \gamma M_m \frac{x_{me}^k}{r_{me}^3} + \gamma \sum_{j=1}^{N} M_j \frac{x_{je}^k}{r_{je}^3} + \frac{1}{M_e} F_e^k \\
\frac{d^2x_m^k}{dt^2} &= \gamma M_s \frac{x_{sm}^k}{r_{ms}^3} + \gamma M_e \frac{x_{em}^k}{r_{me}^3} + \gamma \sum_{j=1}^{N} M_j \frac{x_{jm}^k}{r_{jm}^3} + \frac{1}{M_m} F_m^k
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \qquad \text{(III-5)}$$

Let the subscript c refer to the center of mass of the Earth-Moon system. Thus,

$$\left.\begin{aligned}
M_c &= M_e + M_m \\
x_c^k &= \frac{M_e}{M_c} x_e^k + \frac{M_m}{M_c} x_m^k \\
x_{es}^k &= x_{cs}^k - \frac{M_m}{M_c} x_{me}^k \\
x_{ms}^k &= x_{cs}^k + \frac{M_e}{M_c} x_{me}^k
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \qquad \text{(III-6)}$$

By multiplying the second equation of (III-5) by M_e/M_c and the third equation by M_m/M_c, and by adding the results, we obtain

$$\begin{aligned}
\frac{d^2x_c^k}{dt^2} = \gamma M_s \frac{M_e}{M_c} \frac{x_{se}^k}{r_{es}^3} + \gamma M_s \frac{M_m}{M_c} \frac{x_{sm}^k}{r_{ms}^3} + \gamma \sum_{j=1}^{N} M_j \Bigg(\frac{M_e}{M_c} \frac{x_{je}^k}{r_{je}^3} \\
+ \frac{M_m}{M_c} \frac{x_{jm}^k}{r_{jm}^3}\Bigg) + \frac{1}{M_c}(F_e^k + F_m^k) \quad , \quad k = 1, 2, 3 \quad . \qquad \text{(III-7)}
\end{aligned}$$

By subtracting the first equation of (III-5) from (III-7), we see that the equations of motion of the Earth-Moon barycenter relative to the Sun are

$$\begin{aligned}
\frac{d^2x_{cs}^k}{dt^2} = -\gamma M_s \left(1 + \frac{M_c}{M_s}\right)\left(\frac{M_e}{M_c} \frac{x_{es}^k}{r_{es}^3} + \frac{M_m}{M_c} \frac{x_{ms}^k}{r_{ms}^3}\right) \\
+ \Phi^k + \frac{1}{M_c}(F_e^k + F_m^k) \quad , \quad k = 1, 2, 3 \qquad \text{(III-8)}
\end{aligned}$$

where

$$\Phi^k = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{M_e}{M_c} \frac{x_{je}^k}{r_{je}^3} + \frac{M_m}{M_c} \frac{x_{jm}^k}{r_{jm}^3} - \frac{x_{js}^k}{r_{js}^3} \right) \quad , \quad k = 1, 2, 3 \quad . \tag{III-9}$$

Here we have used the fact that

$$M_e + M_s \frac{M_e}{M_c} = M_s \frac{M_e}{M_c} \left(1 + \frac{M_c}{M_s}\right)$$

$$M_m + M_s \frac{M_m}{M_c} = M_s \frac{M_m}{M_c} \left(1 + \frac{M_c}{M_s}\right) \quad .$$

Next, to obtain the equations of motion of the Moon about the Earth, we subtract the second equation of (III-5) from the third equation, with the result that

$$\frac{d^2 x_{me}^k}{dt^2} = -\gamma M_s \frac{M_c}{M_s} \frac{x_{me}^k}{r_{me}^3} + B^k + \Psi^k + \left(\frac{1}{M_m} F_m^k - \frac{1}{M_e} F_e^k \right) \quad , \quad k = 1, 2, 3 \tag{III-10}$$

where

$$\left.\begin{aligned} B^k &= \gamma M_s \left(\frac{x_{es}^k}{r_{es}^3} - \frac{x_{ms}^k}{r_{ms}^3} \right) \\ \Phi^k &= \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{x_{jm}^k}{r_{jm}^3} - \frac{x_{je}^k}{r_{je}^3} \right) \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{III-11}$$

Effects contained in the $[(1/M_c) \cdot (F_e^k + F_m^k)]$ term in (III-8) determined in this report are

R^k = general relativity effect [see (IV-52)]

S^k = second harmonic of the Sun [see (III-50)] .

Effects contained in the $[(1/M_m) F_m^k - (1/M_e) F_e^k]$ term in (III-10) determined in this report are

H^k = second and third harmonics of the Earth, and the second harmonic of the Moon.

If we let F_e^k and F_m^k denote the forces acting on the Earth and Moon in addition to those enumerated, (III-8) and (III-10) become

$$\left.\begin{aligned} \frac{d^2 x_{cs}^k}{dt^2} &= -\gamma M_s \left(1 + \frac{M_c}{M_s}\right) \left(\frac{M_e}{M_c} \frac{x_{es}^k}{r_{es}^3} + \frac{M_m}{M_c} \frac{x_{ms}^k}{r_{ms}^3} \right) + \Phi^k + R^k \\ &\quad + S^k + \frac{1}{M_c} (F_e^k + F_m^k) \\ \frac{d^2 x_{me}^k}{dt^2} &= -\gamma M_s \frac{M_c}{M_s} \frac{x_{me}^k}{r_{me}^3} + B^k + \Psi^k + H^k + \left(\frac{1}{M_m} F_m^k - \frac{1}{M_e} F_e^k \right) \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{III-12}$$

If we desire the equations of motion of the Earth-Moon barycenter to have the appearance of perturbed elliptic motion, we can write them in the form

$$\frac{d^2 x_{cs}^k}{dt^2} = -\gamma M_s \left(1 + \frac{M_c}{M_s}\right) \frac{x_{cs}^k}{r_{cs}^3} + A^k + \Phi^k + R^k + S^k$$

$$+ \frac{1}{M_c}\left(F_e^k + F_m^k\right) \quad , \quad k = 1, 2, 3 \qquad \text{(III-13)}$$

where

$$A^k = \gamma M_s \left(1 + \frac{M_c}{M_s}\right)\left(\frac{x_{cs}^k}{r_{cs}^3} - \frac{M_e}{M_c}\frac{x_{es}^k}{r_{es}^3} - \frac{M_m}{M_c}\frac{x_{ms}^k}{r_{ms}^3}\right) \quad , \quad k = 1, 2, 3 \quad . \qquad \text{(III-14)}$$

C. PLANETARY PERTURBATIONS

The magnitude of the acceleration relative to the Sun that the Sun gives to planet p is

$$\gamma M_s \left(1 + \frac{M_p}{M_s}\right) \frac{1}{r_{ps}^2} \quad . \qquad \text{(III-15)}$$

The magnitude of the acceleration relative to the Sun that planet j gives to planet p at its closest approach to p (assuming that the Sun, p and j are in a straight line) is

$$\gamma M_s \frac{M_j}{M_s}\left(\frac{1}{r_{jp}^2} \pm \frac{1}{r_{js}^2}\right) \qquad \text{(III-16)}$$

where the plus sign is used if j is between p and the Sun, and the minus sign is used if p is between j and the Sun [see (III-2) and (III-3)]. Using (III-15) and (III-16), the equations in (IV-52) for the general relativity effect, the information in Table I, the fact that $(\gamma M_s) = 2.96 \times 10^{-4}$ AU^3/day^2, and the following discussion of the Earth-Moon dipole term, we derived Table II.

According to (III-3), we can write the effect of the Earth and the Moon on a planet as

$$\Omega_{em}^k = \gamma M_s \frac{M_c}{M_s}\left[\frac{M_e}{M_c}\left(\frac{x_{ep}^k}{r_{ep}^3} - \frac{x_{es}^k}{r_{es}^3}\right) + \frac{M_m}{M_c}\left(\frac{x_{mp}^k}{r_{mp}^3} - \frac{x_{ms}^k}{r_{ms}^3}\right)\right] \quad . \qquad \text{(III-17)}$$

The effect of a hypothetical body of mass $M_c = M_e + M_m$ at the Earth-Moon barycenter is

$$\Omega_c^k = \gamma M_s \frac{M_c}{M_s}\left(\frac{x_{cp}^k}{r_{cp}^3} - \frac{x_{cs}^k}{r_{cs}^3}\right) \quad . \qquad \text{(III-18)}$$

We wish to determine the dipole term

$$T^k = \Omega_{em}^k - \Omega_c^k \quad . \qquad \text{(III-19)}$$

According to (III-6), we can write

TABLE I

PLANETS IN THE SOLAR SYSTEM†

(Note: The masses of the outer planets include those of their satellites.)

Planet	Mass (Sun = 1)	Mean Distance (AU)	Period (years)	Eccentricity	Inclination (deg)
1. Mercury	1.70×10^{-7}	0.387	0.241	0.2056	7.004
2. Venus	2.45×10^{-6}	0.723	0.615	0.0068	3.394
3. Earth-Moon barycenter	3.04×10^{-6}	1.000	1.000	0.0167	0
4. Mars	3.20×10^{-7}	1.524	1.881	0.0934	1.850
5. Jupiter	9.55×10^{-4}	5.203	11.862	0.0484	1.305
6. Saturn	2.85×10^{-4}	9.539	29.458	0.0557	2.490
7. Uranus	4.37×10^{-5}	19.18	84.01	0.0472	0.773
8. Neptune	5.18×10^{-5}	30.06	164.79	0.0086	1.774
9. Pluto	2.78×10^{-6}	39.44	247.69	0.2502	17.170

†See Ref. 5.

TABLE II

MAXIMUM ACCELERATIONS RELATIVE TO THE SUN (AU/DAY^2)

Acc. on \ Due to	Sun	1. Mercury	2. Venus	3. Earth-Moon Barycenter	4. Mars	5. Jupiter	6. Saturn	7. Uranus	8. Neptune	9. Pluto	General Relativity	Earth-Moon Dipole
1. Mercury	1.98×10^{-3}		5.04×10^{-9}	1.50×10^{-9}	3.25×10^{-11}	1.74×10^{-9}	8.07×10^{-11}	1.45×10^{-12}	4.43×10^{-13}	1.03×10^{-14}	1.80×10^{-10}	5.31×10^{-15}
2. Venus	5.66×10^{-4}	7.82×10^{-10}		1.08×10^{-8}	1.06×10^{-10}	3.64×10^{-9}	1.58×10^{-10}	2.80×10^{-12}	8.48×10^{-13}	2.00×10^{-14}	2.35×10^{-11}	1.12×10^{-13}
3. Earth-Moon Barycenter	2.96×10^{-4}	4.70×10^{-10}	1.08×10^{-8}		3.04×10^{-10}	5.56×10^{-9}	2.30×10^{-10}	3.97×10^{-12}	1.18×10^{-12}	2.79×10^{-14}	8.70×10^{-12}	
4. Mars	1.27×10^{-4}	3.75×10^{-10}	2.52×10^{-9}	4.18×10^{-9}		1.04×10^{-8}	3.86×10^{-10}	6.33×10^{-12}	1.86×10^{-12}	4.36×10^{-14}	2.60×10^{-12}	9.35×10^{-15}
5 Jupiter	1.09×10^{-5}	3.38×10^{-10}	1.42×10^{-9}	9.51×10^{-10}	4.78×10^{-11}		3.56×10^{-9}	3.10×10^{-11}	7.85×10^{-12}	1.73×10^{-13}	6.30×10^{-14}	6.59×10^{-16}
6. Saturn	3.25×10^{-6}	3.37×10^{-10}	1.40×10^{-9}	9.12×10^{-10}	4.23×10^{-11}	2.55×10^{-8}		1.04×10^{-10}	1.95×10^{-11}	3.91×10^{-13}	1.05×10^{-14}	6.57×10^{-16}
7. Uranus	8.04×10^{-7}	3.36×10^{-10}	1.39×10^{-9}	9.03×10^{-10}	4.11×10^{-11}	1.19×10^{-8}	1.83×10^{-9}		1.13×10^{-10}	1.48×10^{-12}	1.25×10^{-15}	6.57×10^{-16}
8. Neptune	3.28×10^{-7}	3.36×10^{-10}	1.39×10^{-9}	9.01×10^{-10}	4.09×10^{-11}	1.09×10^{-8}	1.13×10^{-9}	1.45×10^{-10}		8.84×10^{-12}	3.25×10^{-16}	6.57×10^{-16}
9. Pluto	1.90×10^{-7}	3.36×10^{-10}	1.39×10^{-9}	9.00×10^{-10}	4.08×10^{-11}	1.07×10^{-8}	1.02×10^{-9}	6.67×10^{-11}	1.92×10^{-10}		1.45×10^{-16}	6.57×10^{-16}

Note 1. The reason that the maximum accelerations of the outer planets relative to the Sun because of an inner planet vary so little is that an inner planet accelerates the Sun toward the outer planets more than it accelerates the outer planets toward the Sun. A large part (but not all) of the effect of an inner on an outer planet could be accounted for by augmenting the mass of the Sun with the mass of the inner planet.

Note 2. Because of the coordinate system we have used to evaluate the general relativity effect, part of the acceleration due to general relativity advances the perihelion, and part increases the period.

$$r_{es} = r_{cs} \sqrt{1 - 2w_s \frac{M_m}{M_c} \frac{r_{me}}{r_{cs}} + \left(\frac{M_m}{M_c}\right)^2 \left(\frac{r_{me}}{r_{cs}}\right)^2}$$

$$r_{ms} = r_{cs} \sqrt{1 + 2w_s \frac{M_e}{M_c} \frac{r_{me}}{r_{cs}} + \left(\frac{M_e}{M_c}\right)^2 \left(\frac{r_{me}}{r_{cs}}\right)^2} \qquad \text{(III-20)}$$

where

$$w_s = \sum_{\ell=1}^{3} \frac{x_{me}^{\ell}}{r_{me}} \frac{x_{cs}^{\ell}}{r_{cs}} \qquad \text{(III-21)}$$

with exactly similar equations holding with the subscript s replaced by p. According to Ref. 6, we then have

$$\frac{1}{r_{es}} = \frac{1}{r_{cs}} \sum_{\ell=0}^{\infty} P_\ell(w_s) \left(\frac{M_m}{M_c}\right)^\ell \left(\frac{r_{me}}{r_{cs}}\right)^\ell$$

$$\frac{1}{r_{ms}} = \frac{1}{r_{cs}} \sum_{\ell=0}^{\infty} (-1)^\ell P_\ell(w_s) \left(\frac{M_e}{M_c}\right)^\ell \left(\frac{r_{me}}{r_{cs}}\right)^\ell \qquad \text{(III-22)}$$

where the P_ℓ are Legendre polynomials,

$$P_\ell(Z) = \frac{1}{2^\ell} \sum_{i=0}^{[\ell/2]} \frac{(-1)^i (2\ell - 2i)!}{i! (\ell - i)! (\ell - 2i)!} Z^{\ell-2i} \qquad \text{(III-23)}$$

The first few Legendre polynomials are

$$P_0(Z) = 1 \quad , \quad P_1(Z) = Z \quad , \quad P_2(Z) = \frac{3}{2} Z^2 - \frac{1}{2} \quad ,$$

$$P_3(Z) = \frac{5}{2} Z^3 - \frac{3}{2} Z \quad , \quad P_4(Z) = \frac{35}{8} Z^4 - \frac{15}{4} Z^2 + \frac{3}{8} \quad . \qquad \text{(III-24)}$$

From (III-21), we obtain

$$\frac{1}{r_{es}^3} = \frac{1}{r_{cs}^3} \sum_{\ell=0}^{\infty} Q_{3\ell}(w_s) \left(\frac{M_m}{M_c}\right)^\ell \left(\frac{r_{me}}{r_{cs}}\right)^\ell$$

$$\frac{1}{r_{ms}^3} = \frac{1}{r_{cs}^3} \sum_{\ell=0}^{\infty} (-1)^\ell Q_{3\ell}(w_s) \left(\frac{M_e}{M_c}\right)^\ell \left(\frac{r_{me}}{r_{cs}}\right)^\ell \qquad \text{(III-25)}$$

where

$$Q_{3\ell}(Z) = \sum_{\alpha_1+\alpha_2+\alpha_3=\ell} P_{\alpha_1}(Z) P_{\alpha_2}(Z) P_{\alpha_3}(Z) \quad . \qquad \text{(III-26)}$$

According to (III-24), we see that

$$Q_{30}(Z) = 1 \quad , \quad Q_{31}(Z) = 3Z \quad , \quad Q_{32}(Z) = \frac{15}{2} Z^2 - \frac{3}{2} \quad . \qquad \text{(III-27)}$$

Finally, in formula (III-19), by substituting (III-25) and the exactly similar equations obtained by replacing the subscript s by p, and by ignoring powers of r_{me}/r_{cs} and r_{me}/r_{cp} higher than the second, we see that

$$T^k = 3\gamma M_s \frac{M_c}{M_s}\frac{M_e}{M_c}\frac{M_m}{M_c}\left\{\left(\frac{r_{me}}{r_{cs}}\right)^2 \frac{1}{r_{cs}^2}\left[\frac{x^k_{me}}{r_{me}} w_s - \frac{x^k_{cs}}{r_{cs}}\left(\frac{5}{2} w_s^2 - \frac{1}{2}\right)\right]\right.$$

$$\left. - \left(\frac{r_{me}}{r_{cp}}\right)^2 \frac{1}{r_{cp}^2}\left[\frac{x^k_{me}}{r_{me}} w_p - \frac{x^k_{cp}}{r_{cp}}\left(\frac{5}{2} w_p^2 - \frac{1}{2}\right)\right]\right\} \quad . \tag{III-28}$$

The last column in Table II is then calculated using (III-28), Table I and the following data:[5]

Mean distance of Moon from Earth = 384,400 km = 0.0026 AU

$$\frac{M_m}{M_c} = \frac{1}{82.31} = 0.01215$$

$$\frac{M_e}{M_c} = 1 - \frac{M_m}{M_c} = 0.98785 \quad . \tag{III-29}$$

It can be seen that the effect of the Earth and Moon on a planet can be written as (III-18) rather than (III-17), even in the case of Venus, because the entry in Table II represents the maximum magnitude of the dipole term when the planet is closest to the Earth, and this maximum will be much less at other points in the orbit. Further, the sign of the dipole term oscillates as the Moon orbits the Earth.

The largest satellite of Jupiter is Ganymede, and according to Ref. 7 we have

$$\frac{\text{mass of Ganymede}}{\text{mass of Jupiter}} = \frac{1}{12{,}300}$$

$$\text{distance from Jupiter} = 7.156 \times 10^{-3}\,\text{AU} \quad . \tag{III-30}$$

Thus, (III-28) implies that the maximum error in neglecting the displacement of Ganymede from Jupiter in computing the perturbation force on Mars is

$$|T^k| = 1.45 \times 10^{-17}\,\text{AU/day}^2$$

which is three orders of magnitude less than the maximum effect of Pluto on Mars. In general, we can conclude that it is quite accurate to assume that the mass of an outer planet-satellite system is concentrated at the center of mass of the system when determining the effect of the system on a planet.

The total mass of the minor planets (asteroids) is estimated to be about 3/10,000 that of the Earth.[8] It would be rather difficult to include the gravitational effects of these asteroids, except for some of the larger ones, such as Ceres, Pallas, Juno and Vesta. But let us consider the largest asteroid, Ceres. According to Ref. 9, we have

$$\frac{\text{mass of Ceres}}{\text{mass of Sun}} = 3.32 \times 10^{-11}$$

$$\text{mean distance from Sun} = 2.767\,\text{AU} \quad .$$

Thus, by (III-16), the maximum acceleration of Mars due to Ceres is

$$\mathfrak{A} = 5.0 \times 10^{-15}\,\text{AU/day}^2$$

which is less by an order of magnitude than the effect of Pluto on Mars.

The maximum acceleration relative to the Sun that a distant star f can give to a planet p of distance Δ from the Sun is, by (III-16),

$$\mathcal{A}_f = \frac{\gamma M_s (M_f/M_s)}{r_{fs}^2}\left[\left(\frac{r_{fs}}{r_{fp}}\right)^2 - 1\right]$$

$$= \frac{\gamma M_s (M_f/M_s)}{r_{fs}^2}\left[\left(\frac{\Delta}{r_{fp}}\right)^2 + \frac{2\Delta}{r_{fp}}\right]$$

$$\approx \frac{2\gamma M_s (M_f/M_s)\,\Delta}{r_{fs}^3}$$

Here we have assumed that $r_{fs} = r_{fp} + \Delta$. The nearest star is at a distance of 4 light years = 2.52×10^5 AU from the Sun. Assuming that its mass is the same as the Sun's, we have

$$\mathcal{A}_f = 3.70 \times 10^{-20}\,\Delta\,\mathrm{AU/day}^2$$

where Δ ranges between 0.387 AU for Mercury and 39.44 AU for Pluto. To obtain the acceleration of the Moon relative to the Earth due to the star f, we set $\Delta = 0.0026$ AU. Since there are no really massive stars in the neighborhood of the Sun, and since the effect of a star on the acceleration of a planet relative to the Sun goes down as the cube of the distance, we can feel justified in considering the solar system as a closed system in discussing the orbital motions of the planets.

The effect of the displacement of the Moon from the Earth in the equations of motion of the Earth-Moon barycenter is given by the term A^k in (III-13). Inserting (III-25) in formula (III-14) and ignoring powers of (r_{me}/r_{cs}) higher than the second, we obtain

$$A^k = \gamma M_s\left(1 + \frac{M_c}{M_s}\right)\frac{M_e}{M_c}\frac{M_m}{M_c}\left(\frac{r_{me}}{r_{cs}}\right)^2\frac{1}{r_{cs}^2}\left[\frac{x^k_{me}}{r_{me}}\,3w_s - \frac{x^k_{cs}}{r_{cs}}\left(\frac{15}{2}w_s^2 - \frac{3}{2}\right)\right] \quad . \tag{III-31}$$

A simple calculation gives

$$|A^k| \lesssim 2 \times 10^{-10}\,\mathrm{AU/day}^2 \tag{III-32}$$

so that it is important to retain this term in the equations of motion of the Earth-Moon barycenter.

Equations (III-4) for the motion of a planet in the case of a planet with satellites are to be interpreted as the equations of motion of the center of mass of the planet-satellite system. The error in these equations in representing the motion of the center of mass is given by a term similar to the term A^k of (III-14). Mercury and Venus have no detectable satellites, and the satellites of Mars have very small mass; so this possible error is only of concern for the outer planets. However, by (III-30) and (III-31), the error in ignoring the displacement of Ganymede from Jupiter in the equations of motion of the center of mass of the Jovian system satisfies

$$|A^k| \lesssim 10^{-14}\,\mathrm{AU/day}^2 \quad .$$

Further, the short period of revolution of Ganymede about Jupiter (7.15 days) and the long period of revolution of Jupiter about the Sun (11.86 years) would tend to cause the values of A^k at various times to cancel each other. So, in general, we can conclude that (III-4) represents the motion of the center of mass of a planet-satellite system in the case of an outer planet-satellite system.

D. SECOND HARMONIC OF THE GRAVITATIONAL POTENTIAL OF THE SUN

Let (x^1, x^2, x^3) be a Cartesian coordinate system with origin at the center of mass of a body B of mass M. Let dM be an element of mass of the body B. The gravitational potential outside of B is then

$$U = -\gamma \iiint_B \frac{dM(\xi^1, \xi^2, \xi^3)}{[(x^1 - \xi^1)^2 + (x^2 - \xi)^2 + (x^3 - \xi^3)^2]^{1/2}} \quad . \tag{III-33}$$

Let $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. We have

$$\frac{\partial^2 U}{\partial (x^1)^2} + \frac{\partial^2 U}{\partial (x^2)^2} + \frac{\partial^2 U}{\partial (x^3)^2} = 0$$

$$U \approx -\frac{\gamma M}{r} \quad \text{for large } r \quad . \tag{III-34}$$

We introduce spherical coordinates (r, Θ, φ) by the formulas

$$\left.\begin{aligned} x^1 &= r \sin\Theta \cos\varphi \\ x^2 &= r \sin\Theta \sin\varphi \\ x^3 &= r \cos\Theta \end{aligned}\right\} \quad 0 \leqslant r < \infty, \ 0 \leqslant \varphi < 2\pi, \ 0 \leqslant \Theta \leqslant \pi \quad . \tag{III-35}$$

Because of (III-34), U can be expanded in spherical harmonics

$$U = -\frac{\gamma M}{r} + \frac{\gamma M}{r} \sum_{n=1}^{\infty} \sum_{h=0}^{n} (a_{nh} \cos h\varphi + b_{nh} \sin h\varphi) \frac{P_{nh}(\cos\Theta)}{r^n} \tag{III-36}$$

where

$$P_{no}(Z) = P_n(Z) = \frac{1}{2^n n!} \frac{d^n}{dZ^n} (Z^2 - 1)^n \qquad n = 0, 1, 2, \ldots$$

$$P_{nh}(Z) = (1 - Z^2)^{h/2} \frac{d^h P_n(Z)}{dZ^h} \qquad h = 0, 1, \ldots, n \tag{III-37}$$

(see Refs. 10 and 11). The first few Legendre polynomials $P_n(Z)$ are given in (III-24).

Let us write

$$r(x - \xi) = \left[\sum_{j=1}^{3} (x^j - \xi^j)^2\right]^{1/2}$$

$$r = \left[\sum_{j=1}^{3} (x^j)^2\right]^{1/2}$$

$$\xi = \left[\sum_{j=1}^{3} (\xi^j)^2\right]^{1/2} \tag{III-38}$$

According to Ref. 6, we then have

$$\frac{1}{r(x-\xi)} = \frac{1}{r} \sum_{\ell=0}^{\infty} P_{\ell}(q) \left(\frac{\xi}{r}\right)^{\ell} \tag{III-39}$$

where q is the cosine of the angle between the vectors $\vec{r}$ and $\vec{\xi}$,

$$q = \sum_{j=1}^{3} \frac{x^j}{r} \frac{\xi^j}{\xi} \quad . \tag{III-40}$$

By (III-33) this implies that

$$U = -\frac{\gamma M}{r} - \frac{\gamma}{r} \sum_{n=1}^{\infty} \frac{1}{r^n} \iiint_B P_n(q)\, \xi^n dM(\xi) \quad . \tag{III-41}$$

Comparing (III-41) with formula (III-36), we see that

$$M \sum_{h=0}^{n}{}' (a_{nh} \cos h\varphi + b_{nh} \sin h\varphi)\, P_{nh}(\cos\Theta) = -\iiint_B P_n(q)\, \xi^n dM(\xi) \quad . \tag{III-42}$$

Since the origin of the coordinate system is at the center of mass of the body, we have

$$\iiint_B P_1(q)\, \xi dM(\xi) = -\sum_{j=1}^{3} \frac{x^j}{r} \iiint_B \xi^j dM(\xi) = 0 \quad . \tag{III-43}$$

Thus, the summation in (III-36) can start with n = 2,

$$U = -\frac{\gamma M}{r} + \frac{\gamma M}{r} \sum_{n=2}^{\infty} \sum_{h=0}^{n} (a_{nh} \cos h\varphi + b_{nh} \sin h\varphi) \frac{P_{nh}(\cos\varphi)}{r^n} \quad . \tag{III-44}$$

If the body is symmetric about a line through its center of mass, and we choose the x^3-axis to point along this line, (III-44) reduces to

$$U = -\frac{\gamma M}{r} + \frac{\gamma M}{r} \sum_{n=2}^{\infty} \frac{J_n}{r^n} P_n(\cos\Theta) \tag{III-45}$$

where we have written J_n for a_{no}. In the case of the Sun, we thus suppose that the gravitational potential is given by

$$U = -\frac{\gamma M_s}{r} + \frac{\gamma M_s}{r} \left(\frac{R_s}{r}\right)^2 \left(\frac{S_2}{R_s}\right)^2 \left[\frac{3}{2}\left(\frac{x^3}{r}\right)^2 - \frac{1}{2}\right] \tag{III-46}$$

where $R_s = 6.96 \times 10^5$ km is the radius of the Sun, and the coefficient S_2 of the second harmonic of the Sun is to be determined by its effect on the motion of a planet. We may assume that the x^3-axis of symmetry of the Sun points along the axis of rotation of the Sun.

Let (X^1, X^2, X^3) be the coordinate system with origin at the center of mass of the Sun in which the equations of motion are expressed. We have

$$\left.\begin{aligned} x^j &= \sum_{\ell=1}^{3} C_{j\ell} X^\ell \\ X^j &= \sum_{\ell=1}^{3} C_{\ell j} x^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{III-47}$$

where $(C_{j\ell})$ is an orthogonal matrix. Let (F^1, F^2, F^3) be the components of the force on a planet of mass M_p due to the Sun in the (X^1, X^2, X^3) coordinate system. Since

$$\vec{F} = -M_p \, \overrightarrow{\text{grad } U} \tag{III-48}$$

formulas (III-46) and (III-47) give

$$F^k = -\frac{\gamma M_s M_p X^k}{r^3} + \frac{\gamma M_s M_p}{r^2}\left(\frac{R_s}{r}\right)^2 \frac{S_2}{R_s^2} \cdot \left\{\frac{X^k}{r}\left[\frac{15}{2}\left(\sum_{\ell=1}^{3} C_{3\ell}\frac{X^\ell}{r}\right)^2 - \frac{3}{2}\right]\right.$$

$$\left. -3\left(\sum_{\ell=1}^{3} C_{3\ell}\frac{X^\ell}{r}\right)C_{3k}\right\} , \quad k=1, 2, 3 \quad . \tag{III-49}$$

The force on the Sun due to the planet is minus (III-49). Since the equations in (III-4) are derived by subtracting the equations of motion of the Sun from the equations of motion of the planet, the term S^k in (III-4) is [for use in (III-12) the subscript p should be replaced by c]

$$S^k = \frac{\gamma M_s \left(1 + \frac{M_p}{M_s}\right)}{r_{ps}^2}\left(\frac{R_s}{r_{ps}}\right)^2 \frac{S_2}{R_s^2}\left[\frac{x_{ps}^k}{r_{ps}}\left(\frac{15}{2} g^2 - \frac{3}{2}\right) - 3gC_{3k}\right] , \quad k = 1, 2, 3 \tag{III-50}$$

where $x_{ps}^k = X^k$, and where

$$g = \sum_{\ell=1}^{3} C_{3\ell}\frac{x_{ps}^\ell}{r_{ps}} \quad . \tag{III-51}$$

We recall that S_2 is the second harmonic of the Sun's gravitational potential and has the dimensions of a length squared, and that R_s is the equatorial radius of the Sun. The quantities $C_{3\ell}$ $(\ell = 1, 2, 3)$ are determined in Appendix C.

E. HIGHER HARMONICS IN THE GRAVITATIONAL POTENTIALS OF THE EARTH AND MOON

The purpose of this section is to derive the force on the Moon due to the Earth, considering terms up to the third harmonic in the Earth's gravitational potential and up to the second harmonic in the Moon's gravitational potential.

We shall assume that the Earth is symmetric about its axis of rotation. Let the coordinate system (x^1, x^2, x^3), with origin at the center of mass of the Earth, be referred to the true equinox and equator of date, so that the x^3-axis points along this axis of rotation. Then, by (III-45) and (III-24), we have

$$U = -\frac{\gamma M_e}{r} + \frac{\gamma M_e J_2}{r^3}\left(\frac{3}{2}\cos^2\Theta - \frac{1}{2}\right) + \frac{\gamma M_e J_3}{r^4}\left(\frac{5}{2}\cos^3\Theta - \frac{3}{2}\cos\Theta\right) \tag{III-52}$$

where, by (III-35),

$$\cos\Theta = \frac{x^3}{r} \quad . \tag{III-53}$$

According to Ref. 12, we have

$$\frac{J_2}{R_e^2} = 1.0827 \times 10^{-3}$$

$$\frac{J_3}{R_e^3} = -2.4 \times 10^{-6} \tag{III-54}$$

where $R_e = 6378.17$ km is the equatorial radius of the Earth.

Let (X^1, X^2, X^3) be the coordinate system referred to the mean equinox and equator of 1950.0, the reference system in which we are integrating the equations of motion. Choosing the origin of this coordinate system to be at the center of mass of the Earth, we can write

$$\left.\begin{aligned} x^j &= \sum_{\ell=1}^{3} A_{j\ell} X^\ell \\ X^j &= \sum_{\ell=1}^{3} A_{\ell j} x^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{III-55}$$

where the orthogonal matrix $(A_{j\ell})$ is given in Appendix A. Then, by (III-52) through (III-55) and (III-48), the components of force (F^1, F^2, F^3) on a particle of mass M due to the Earth in the coordinate system (X^1, X^2, X^3) are

$$\begin{aligned} F^k(X) = &-\frac{\gamma M_e M X^k}{r^3} + \frac{\gamma M_e M J_2}{r^4}\left[\frac{X^k}{r}\left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right) - 3A_{3k}\cos\Theta\right] \\ &+ \frac{\gamma M_e M J_3}{r^5}\left[\frac{X^k}{r}\left(\frac{35}{2}\cos^3\Theta - \frac{15}{2}\cos\Theta\right)\right. \\ &\left. - A_{3k}\left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right)\right] \quad , \quad k = 1, 2, 3 \quad . \end{aligned} \tag{III-56}$$

Suppose that B is an extended physical body of mass M (the Moon in our case), and let (X^1, X^2, X^3) be the coordinates of its center of mass. Let (ξ^1, ξ^2, ξ^3) be the coordinates of a mass element dM in B relative to the center of mass of B. By (III-56), the force $\vec{F}_B$ on B due to the gravitational field of the Earth has components

$$F_B^k(X) = \iiint_B F^k(X + \xi)\,\frac{dM(\xi)}{M} = F^k(X) + C^k \tag{III-57}$$

where

$$C^k = \iiint_B [F^k(X + \xi) - F^k(X)] \frac{dM(\xi)}{M} \quad .$$

We write

$$C^k = C_1^k + C_2^k + C_3^k \tag{III-58}$$

where

$$C_1^k = -\gamma M_e \iiint_B \left[\frac{X^k + \xi^k}{r(X + \xi)^3} - \frac{X^k}{r(X)^3}\right] dM(\xi)$$

$$C_2^k = \gamma M_e J_2 \iiint_B [G^k(X + \xi) - G^k(X)] \; dM(\xi)$$

$$C_3^k = \gamma M_e J_3 \iiint_B [H^k(X + \xi) - H^k(X)] \; dM(\xi) \tag{III-59}$$

where

$$r(X - \xi) = \left[\sum_{j=1}^{3} (X^j + \xi^j)^2\right]^{1/2}$$

$$G^k(X) = \frac{X^k}{r^5} \left(\frac{15}{2} \cos^2 \Theta - \frac{3}{2}\right) - \frac{3A_{3k} \cos \Theta}{r^4}$$

$$H^k(X) = \frac{X^k}{r^6} \left(\frac{35}{2} \cos^3 \Theta - \frac{15}{2} \cos \Theta\right) - \frac{A_{3k}}{r^5} \left(\frac{15}{2} \cos^2 \Theta - \frac{3}{2}\right) \quad . \tag{III-60}$$

Equation (III-39) implies that

$$\frac{1}{r(X + \xi)} = \frac{1}{r} \sum_{\ell=0}^{\infty} (-1)^\ell \, P_\ell(q) \left(\frac{\xi}{r}\right)^\ell \tag{III-61}$$

where the $P_\ell(q)$ are Legendre polynomials and where

$$r = \left[\sum_{j=1}^{3} (X^j)^2\right]^{1/2}$$

$$\xi = \left[\sum_{j=1}^{3} (\xi^j)^2\right]^{1/2}$$

$$q = \sum_{j=1}^{3} \frac{X^j}{r} \frac{\xi^j}{\xi} \quad . \tag{III-62}$$

Since $P_\ell(q)$ only contains even powers of q for ℓ even, and odd powers of q for ℓ odd, expression (III-61) contains no square roots of the quantities (ξ^1, ξ^2, ξ^3), only products and powers. In fact, $P_\ell(q)\, \xi^\ell$ is a homogeneous polynomial in (ξ^1, ξ^2, ξ^3) of degree ℓ with coefficients depending on $(X^1/r,\ X^2/r,\ X^3/r)$. Equation (III-61) implies that

$$\frac{1}{r(X+\xi)^n} = \frac{1}{r^n} = \sum_{\ell=0}^{\infty} (-1)^\ell Q_{n\ell}(q)\xi^\ell \frac{1}{r^\ell}$$

$$Q_{n\ell}(q) = \sum_{\alpha_1+\ldots+\alpha_n=\ell} P_{\alpha_1}(q)\ldots P_{\alpha_n}(q) \qquad \text{(III-63)}$$

where $Q_{n\ell}(q)\xi^\ell$ is a homogeneous polynomial in (ξ^1, ξ^2, ξ^3) of degree ℓ with coefficients depending on $(X^1/r,\ X^2/r,\ X^3/r)$.

If we insert the above expression for $1/r(X+\xi)^n$ into the integrands in (III-59), we obtain

$$C_1^k = \frac{\gamma M_e}{r^2} \sum_{\ell=0}^{\infty} \frac{1}{r^\ell} \iiint_B R_{1\ell}^k(\xi, \frac{X}{r})\, dM(\xi)$$

$$C_2^k = \frac{\gamma M_e J_2}{r^4} \sum_{\ell=0}^{\infty} \frac{1}{r^\ell} \iiint_B R_{2\ell}^k(\xi, \frac{X}{r})\, dM(\xi)$$

$$C_3^k = \frac{\gamma M_e J_3}{r^5} \sum_{\ell=0}^{\infty} \frac{1}{r^\ell} \iiint_B R_{3\ell}^k(\xi, \frac{X}{r})\, dM(\xi) \qquad \text{(III-64)}$$

where the $R_{i\ell}^k\ [\xi, (X/r)]$ are homogeneous polynomials in $(\xi^1,\ \xi^2,\ \xi^3)$ of degree ℓ with coefficients depending on $(X^1/r,\ X^2/r,\ X^3/r)$. Actually, the above series start with $\ell = 1$, because the integrands in (III-59) are the differences of functions evaluated at $(X + \xi)$ and X. Since $(\xi^1,\ \xi^2,\ \xi^3) = (0,\ 0,\ 0)$ is the center of mass of the body,

$$\iiint_B \xi^j dM(\xi) = 0 \quad , \quad j = 1, 2, 3 \qquad \text{(III-65)}$$

so we may assume that the series (III-64) start with $\ell = 2$. The integrals $\iiint_B R_{i2}^k\, dM(\xi)$ involve the moments and products of inertia of the body, while the integrals $\iiint_B R_{i\ell}^k\, dM(\xi)\ (\ell > 2)$ depend on the higher moments of the body. We ignore these higher moments which, by the discussion in Ref. 13, is equivalent to ignoring harmonics higher than the second in the body's (= Moon's) gravitational potential. We can therefore assume that

$$C_1^k = \frac{\gamma M_e}{r^4} \iiint_B R_{12}^k(\xi, \frac{X}{r})\, dM(\xi)$$

$$C_2^k = \frac{\gamma M_e J_2}{r^6} \iiint_B R_{22}^k(\xi, \frac{X}{r})\, dM(\xi)$$

$$C_3^k = \frac{\gamma M_e J_3}{r^7} \iiint_B R_{32}^k(\xi, \frac{X}{r})\, dM(\xi) \quad . \qquad \text{(III-66)}$$

The effect of the third harmonic of the Earth's gravitational potential in the force $F^k(X)$ of (III-56) is of the order J_3/r^5. Thus, to the accuracy to which we are working in this section, we can assume that C_2^k and C_3^k are zero, because the coefficients multiplying $1/r^6$ and $1/r^7$ in C_2^k and C_3^k will involve J_2 and J_3 times similar constants associated with the second harmonic

of the Moon's gravitational potential. (The fact that C_2^k and C_3^k are no larger than the effect of the fourth harmonic of the Earth was actually checked by direct computation, but the calculations are too lengthy to reproduce in this report.) Finally, examining expression (III-59) for C_1^k, and noting (III-33) and (III-48), we may assume that C_1^k is minus the force due to the second harmonic of the Moon's gravitational potential acting on the Earth (where the Earth is considered to be a point mass).

The Moon is approximately an ellipsoid with three unequal axes. Let (x^1, x^2, x^3) be a coordinate system with origin at the center of mass of the Moon such that the x^3-axis points toward the north pole of the Moon (which is one of the axes of the ellipsoid), the x^1-axis points along the axis of the ellipsoid pointed in the direction of the Earth, and the x^2-axis completes the right-hand system. Let I_j be the moment of inertia of the Moon with respect to the x^j-axis. We may assume that the products of inertia with respect to the (x^1, x^2, x^3) frame are zero, so that

$$\left.\begin{aligned} \iiint_B \left[\sum_{\ell \neq j} (\xi^\ell)^2\right] dM(\xi) &= I_j \quad , \qquad j = 1, 2, 3 \\ \iiint_B \xi^i \xi^j \, dM(\xi) &= 0 \quad , \qquad i \neq j \end{aligned}\right\} \quad . \tag{III-67}$$

Let a, b, c be the axes of the ellipsoidal Moon in the x^1, x^2, x^3 directions. According to Ref. 14, we have

$$\begin{aligned} \frac{b+c}{2} &= 1737.9 \text{ km} \\ a - c &= 1.09 \text{ km} \\ a - b &= 0.36 \text{ km} \quad . \end{aligned} \tag{III-68}$$

$$\begin{aligned} \frac{I_3}{b^2 M_m} &= 0.397 \\ \frac{I_3 - I_2}{I_1} &= 0.000420 \\ \frac{I_3 - I_1}{I_2} &= 0.000628 \\ \frac{I_2 - I_1}{I_3} &= 0.000208 \quad . \end{aligned} \tag{III-69}$$

The above values of the moments of inertia of the Moon (which determine the second harmonic of the Moon's gravitational potential) were obtained from the observed shape and physical libration of the Moon. It is to be expected that in the near future the second and higher harmonics of the Moon's gravitational potential will be accurately determined by placing an artificial satellite in orbit about the Moon.

According to (III-24), (III-42) and (III-44), the second harmonic of the Moon's gravitational potential is

$$U_2 = -\frac{\gamma}{r^3} \iiint P_2(q)\, \xi^2 dM(\xi)$$

$$= -\frac{\gamma}{r^3} \left[\frac{3}{2} \sum_{i,j} \frac{x^i x^j}{r^2} \iiint \xi^i \xi^j dM(\xi) - \frac{1}{2} \sum_j \iiint (\xi^j)^2 dM(\xi) \right] .$$

Equations (III-67) give

$$U_2 = \frac{\gamma}{2r^3} \sum_{j=1}^{3} \left[1 - 3\left(\frac{x^j}{r}\right)^2 \right] \iiint (\xi^j)^2 dM(\xi)$$

$$= \frac{\gamma}{4r^3} \sum_{j=1}^{3} \left[1 - 3\left(\frac{x^j}{r}\right)^2 \right] \left(\sum_{\ell \neq j} I_\ell - I_j \right) .$$

Using the fact that $\sum_{j=1}^{3} (x^j/r)^2 = 1$, this can be put in the form

$$U_2 = \frac{\gamma}{r^3} \sum_{j=1}^{3} \left[\frac{3}{2} \left(\frac{x^j}{r}\right)^2 - \frac{1}{2} \right] I_j$$

$$= \frac{\gamma}{r^3} \sum_{j=2}^{3} \left[\frac{3}{2} \left(\frac{x^j}{r}\right)^2 - \frac{1}{2} \right] (I_j - I_1) \quad . \qquad \text{(III-70)}$$

Introducing polar coordinates (III-35), the first line of (III-70) implies that

$$U_2 = \frac{\gamma}{r^3} \left[\frac{3}{2}(I_1 \cos^2\varphi + I_2 \sin^2\varphi) \sin^2\Theta + \frac{3}{2} I_3 \cos^2\Theta - \frac{1}{2}(I_1 + I_2 + I_3) \right]$$

$$= \frac{\gamma}{r^3} \left\{ \left[I_3 - \frac{1}{2}(I_1 + I_2) \right] \left[\frac{3}{2} \cos^2\Theta - \frac{1}{2} \right] + \frac{3}{4}(I_1 - I_2) \cos 2\varphi \sin^2\Theta \right\} \quad . \qquad \text{(III-71)}$$

Comparing this expression with (III-37), (III-41) and (III-42), we see that

$$U_2 = \frac{\gamma M_m}{r^3} \left[a_{20} P_{20}(\cos\Theta) + a_{22} \cos 2\varphi\, P_{22}(\cos\Theta) \right] \qquad \text{(III-72)}$$

where

$$M_m a_{20} = I_3 - \frac{1}{2}(I_1 + I_2)$$

$$M_m a_{22} = \frac{1}{4}(I_1 - I_2) \qquad \text{(III-73)}$$

which implies that

$$I_2 - I_1 = 4M_m a_{22}$$

$$I_3 - I_1 = M_m(a_{20} - 2a_{22}) \quad . \qquad \text{(III-74)}$$

We shall use expression (III-70) for the second harmonic of the Moon's gravitational potential. We have derived (III-72) through (III-74) so that we can determine improved values of the

quantities $(I_2 - I_1)$ and $(I_3 - I_1)$ in (III-70) from a possible future publication of improved values of a_{20} and a_{22}.

Let (X^1, X^2, X^3) be the coordinates of the center of mass of the Moon relative to the center of mass of the Earth in the coordinate system in which we are integrating the equations of motion. Let (x^1, x^2, x^3) be the coordinates of the center of mass of the Earth relative to the center of mass of the Moon in the coordinate system used in formula (III-70). Then, the relation between (X^1, X^2, X^3) and (x^1, x^2, x^3) is given by

$$\left.\begin{aligned} x^j &= -\sum_{\ell=1}^{3} B_{j\ell} X^\ell \\ X^j &= -\sum_{\ell=1}^{3} B_{\ell j} x^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{III-75}$$

where the orthogonal matrix $B_{\ell j}$ is determined in Appendix B.

Let C_1^k be the components in the (X^1, X^2, X^3) coordinate system of minus the force due to the second harmonic of the Moon's gravitational potential acting on the Earth (where the Earth is considered to be a point mass). Then, by (III-48), (III-70) and (III-75), we have

$$\begin{aligned} C_1^k &= -M_e \frac{\partial U_2}{\partial X^k} \\ &= \frac{\gamma M_e}{r^4} \sum_{j=2}^{3} (I_j - I_1) \left[\frac{X^k}{r}\left(\frac{15}{2} D_j^2 - \frac{3}{2}\right) - 3D_j B_{jk}\right] \end{aligned} \tag{III-76}$$

where

$$D_j = -\frac{x^j}{r} = \sum_{\ell=1}^{3} B_{j\ell} \frac{X^\ell}{r} \quad . \tag{III-77}$$

Formulas (III-56), (III-57), (III-58) and (III-76) combine to give that the force on the Moon due to the Earth is

$$\begin{aligned} F_m^k = &-\frac{\gamma M_e M_m X^k}{r^3} + \frac{\gamma M_e M_m}{r^2}\left(\frac{R_e}{r}\right)^2 \frac{J_2}{R_e^2}\left[\frac{X^k}{r}\left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right) - 3A_{3k}\cos\Theta\right] \\ &+ \frac{\gamma M_e M_m}{r^2}\left(\frac{R_m}{r}\right)^2 \frac{I_3}{M_m R_m^2} \sum_{j=2}^{3} \frac{I_j - I_1}{I_3}\left[\frac{X^k}{r}\left(\frac{15}{2} D_j^2 - \frac{3}{2}\right) - 3D_j B_{jk}\right] \\ &+ \frac{\gamma M_e M_m}{r^2}\left(\frac{R_e}{r}\right)^3 \frac{J_3}{R_e^3}\left[\frac{X^k}{r}\left(\frac{35}{2}\cos^3\Theta - \frac{15}{2}\cos\Theta\right)\right. \\ &\left. - A_{3k}\left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right)\right] \quad , \quad k = 1, 2, 3 \end{aligned} \tag{III-78}$$

where R_e and R_m are the radii of the Earth and Moon, and where we have assumed that $C_2^k = 0$, $C_3^k = 0$. The force F_e^k on the Earth due to the Moon is minus the force on the Moon due to the

Earth, $F_e^k = -F_m^k$. Equations (III-10) for the motion of the Moon relative to the Earth were derived by subtracting the equations of motion of the Earth from the equations of motion of the Moon. Thus, the H^k term in (III-12) is

$$H^k = \frac{\gamma M_s}{r_{me}^2} \frac{M_c}{M_s} \left\{ \left(\frac{R_e}{r_{me}}\right)^2 \frac{J_2}{R_e^2} \left[\frac{x_{me}^k}{r_{me}} \left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right) - 3A_{3k}\cos\Theta \right] \right.$$

$$+ \left(\frac{R_m}{r_{me}}\right)^2 \frac{I_3}{M_m R_m^2} \sum_{j=2}^{3} \frac{I_j - I_1}{I_3} \left[\frac{x_{me}^k}{r_{me}} \left(\frac{15}{2} D_j^2 - \frac{3}{2}\right) - 3D_j B_{jk} \right]$$

$$+ \left(\frac{R_e}{r_{me}}\right)^3 \frac{J_3}{R_e^3} \left[\frac{x_{me}^k}{r_{me}} \left(\frac{35}{2}\cos^3\Theta - \frac{15}{2}\cos\Theta\right) \right.$$

$$\left.\left. -A_{3k}\left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right)\right]\right\} \quad , \quad k = 1, 2, 3 \qquad \text{(III-79)}$$

where we have used the notation of Sec. III-B with $x_{me}^k = X^k$, $r_{me} = r$, and $M_c = M_e + M_m$. The constants (R_e, J_2, J_3) and (R_m, I_1, I_2, I_3) are given in (III-54), (III-68) and (III-69). (We must evidently assume that $R_m = b$.) The matrices (A_{ij}) and (B_{ij}) are determined in Appendices A and B. By (III-53) and (III-77), the quantities $\cos\Theta$ and D_j in (III-79) are

$$\cos\Theta = \sum_{\ell=1}^{3} A_{3\ell} \frac{x_{me}^{\ell}}{r_{me}}$$

$$D_j = \sum_{\ell=1}^{3} B_{j\ell} \frac{x_{me}^{\ell}}{r_{me}} \quad , \quad j = 2, 3 \quad . \qquad \text{(III-80)}$$

Table III is constructed using the expressions for the forces perturbing the motion of the Moon relative to the Earth given in (III-11), (III-12) and (III-79).

TABLE III

MAXIMUM ACCELERATIONS OF THE MOON RELATIVE TO THE EARTH (AU/DAY^2)
(Note: A constant acceleration of 10^{-15} AU/day^2 will move a body 1 km in 10 years.)

Due to	Acceleration	Due to	Acceleration
Earth	1.33×10^{-4}	Mars	3.45×10^{-12}
Sun	1.55×10^{-6}	Jupiter	1.98×10^{-11}
2nd harmonic of Earth	2.40×10^{-10}	Saturn	7.05×10^{-13}
2nd harmonic of Moon	3.00×10^{-12}	Uranus	1.12×10^{-14}
3rd harmonic of Earth	2.00×10^{-14}	Neptune	3.25×10^{-15}
Mercury	1.14×10^{-12}	Pluto	7.53×10^{-17}
Venus	1.80×10^{-10}		

The higher harmonics of the Earth's gravitational potential are known quite accurately because of their effects on the motions of artificial Earth satellites. If the higher harmonics of the Moon's gravitational potential are similarly determined by placing an artificial satellite in orbit about the Moon, the effect of both the second and third harmonics of the Moon can be included in the H^k term above. In this case, the second, third and fourth harmonics of the Earth, the interaction between the second harmonics of the Earth and Moon [the C_2^k term in (III-66)], and the effect of the terms in the gravitational potential of the Earth which arise from asymmetries about the north-south axis (tesseral harmonics) can all be included.

The effect of tidal friction on the motion of the Moon is small, since it is estimated that the increase in the sidereal day as a result of tidal action is 7.2×10^{-4} sec per century.[15] However, the effect of tidal friction should be included in the equations of motion of the Moon, if harmonics in the gravitational potentials of the Earth and Moon higher than those considered in this report are included in the equations of motion.

IV. GENERAL RELATIVISTIC EFFECT

A. MATHEMATICAL FORMALISM OF EINSTEIN'S GENERAL THEORY OF RELATIVITY

Euclidean n-space $\mathfrak{R}^n$ is defined as the set of all n-tuples $(x^1, \ldots, x^n)$ of real numbers x^j $(j = 1, \ldots, n)$. An open ball of radius r about a point $x_o = (x_o^1, \ldots, x_o^n)$ in $\mathfrak{R}^n$ is the set of all points $x = (x^1, \ldots, x^n)$ in $\mathfrak{R}^n$ such that $\sqrt{\sum_{j=1}^{n} (x^j - x_o^j)^2} < r$. A manifold M^n of dimension n is a separable connected Hausdorff space such that each point in M^n has a neighborhood which is homeomorphic to an open ball in $\mathfrak{R}^n$.† That is, each point p in M^n has a neighborhood U in M^n such that there is a one-to-one bicontinuous map $f:U \to B$ onto an open ball B in $\mathfrak{R}^n$. The map f puts a coordinate system on U in the sense that to each point q in U is associated the coordinates $(x^1, \ldots, x^n) = f(q)$. Suppose that $f_1:U_1 \to B_1$ and $f_2:U_2 \to B_2$ are two coordinate systems such that the intersection of U_1 and U_2 is not empty. Then the change of coordinates is given by the map $f_2 \circ f_1^{-1}:B_1 \to B_2$, as sketched in Fig 2. A differentiable manifold of dimension n is a manifold of dimension n which has a covering by coordinate neighborhoods such that the coordinate transformations are infinitely differentiable. The manifold is analytic if the coordinate transformations are analytic.

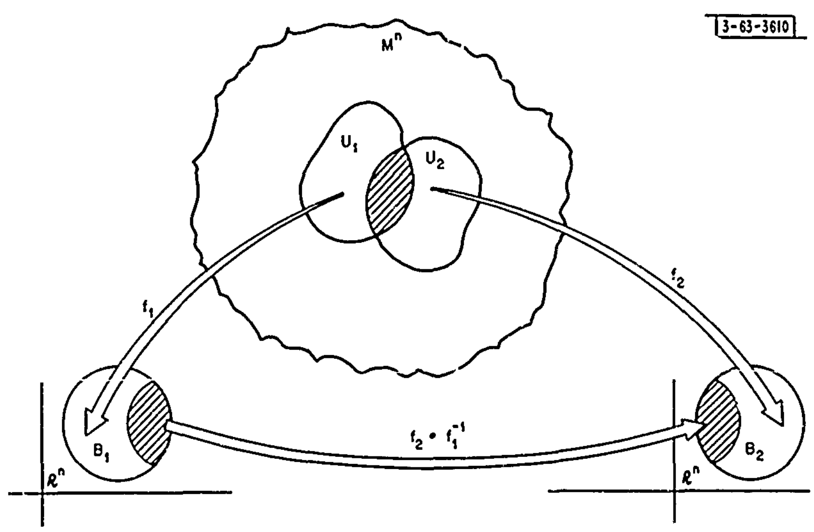

Fig. 2. Coordinate systems on a manifold.

The simplest example of an n-dimensional manifold is Euclidean n-space itself. Examples of two dimensional manifolds are provided by surfaces in Euclidean three-space, such as the cylinder, torus and sphere. A manifold can be defined without any reference to a higher dimensional Euclidean space. Roughly speaking, one might imagine that a manifold is a space which can curve back on itself in the large, but which locally looks like Euclidean space.

Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be two overlapping coordinate systems on a differentiable manifold M^n of dimension n. A tensor T contravariant of order p and covariant of order q is expressed in these two coordinate systems in the form‡

---

†See Ref. 16 for definitions of these topological concepts.

‡See Ref. 17 for a rigorous abstract definition of tensors on a differentiable manifold.

$$T = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} dx^{\nu_1} \otimes \dots \otimes dx^{\nu_q} \otimes \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_p}}$$

$$T = S^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} dy^{\beta_1} \otimes \dots \otimes dy^{\beta_q} \otimes \frac{\partial}{\partial y^{\alpha_1}} \otimes \frac{\partial}{\partial y^{\alpha_p}} \quad . \tag{IV-1}$$

Here, we use the Einstein summation convention in which repeated upper and lower indices are summed. The components of the tensor T transform according to the tensor rule of transformation:

$$S^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} \dots \frac{\partial x^{\nu_q}}{\partial y^{\beta_q}} \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_p}}{\partial x^{\mu_p}} \quad . \tag{IV-2}$$

In the general theory of relativity, the space-time universe is imagined to be a four-dimensional differentiable (or perhaps analytic) manifold. The gravitational potential in the space-time universe is given by a symmetric hyperbolic covariant tensor of order two $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$, called the metric tensor. Symmetric means that $g_{\mu\nu} = g_{\nu\mu}$, and hyperbolic means that for each point of space-time universe there is a coordinate system (x^0, x^1, x^2, x^3) such that

$$ds^2 = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3 \tag{IV-3}$$

at that point.

Let $(g^{\alpha\beta})$ denote the matrix inverse to the matrix $(g_{\mu\nu})$. The Christoffel symbols are then defined by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right) \tag{IV-4}$$

(see Ref. 18). If we think of the metric tensor $g_{\mu\nu}$ as being the gravitational potential, then we should think of the Christoffel symbols as being the gravitational field. The Riemann curvature tensor is defined by

$$R^{\alpha}_{\mu\nu\beta} = \frac{\partial \Gamma^{\alpha}_{\mu\beta}}{\partial x^\nu} - \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^\beta} + \Gamma^{\lambda}_{\mu\beta} \Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu} \Gamma^{\alpha}_{\lambda\beta} \quad . \tag{IV-5}$$

We further define

$$R_{\mu\nu} = R^{\beta}_{\mu\nu\beta}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{IV-6}$$

(see Ref. 19).

It is then postulated that the gravitational potential in the space-time universe satisfies the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{IV-7}$$

where κ is a constant and where $T_{\mu\nu}$ is a tensor defined in terms of the distribution of matter and energy in the space-time universe. Multiplying both sides of this equation by $g^{\mu\nu}$, and summing on μ and ν, we see that

$$R = \kappa T \quad . \tag{IV-8}$$

If there is no matter at a given point of the space-time universe, then $T_{\mu\nu} = 0$ and by (IV-7) and (IV-8) the Einstein field equations become

$$R_{\mu\nu} = 0 \tag{IV-9}$$

at this point.

The above equations involving the metric, Christoffel symbols and Riemannian curvature can be expressed in more abstract differential-geometric terms. This abstract approach would be appropriate for a discussion of the space-time universe in the large. For local discussions, the formulation in terms of local coordinate systems is sufficient.

A curve, or world line, in the space-time universe M is a map γ: $[a, b] \rightarrow M$ of an interval $[a, b]$ in the real numbers into M (see Fig. 3). In a coordinate system (x^0, x^1, x^2, x^3) on a coordinate neighborhood U in M, the curve can be written in the form

$$x^\mu = \gamma^\mu(s) \quad , \quad s \in [a, b] \quad . \tag{IV-10}$$

The tangent vector to the curve is then

$$\lambda = \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu} \quad . \tag{IV-11}$$

A vector $\lambda = \lambda^\mu(\partial/\partial x^\mu)$ at a point is said to be time-like if $g_{\mu\nu}\lambda^\mu\lambda^\nu > 0$, null if $g_{\mu\nu}\lambda^\mu\lambda^\nu = 0$, and space-like if $g_{\mu\nu}\lambda^\mu\lambda^\nu < 0$. The path of a light ray through the space-time universe has null tangent vector, while the path of a material body through the space-time universe has time-like tangent vector. A curve in the space-time universe with space-like tangent vector has no physical interpretation.

An observer in the space-time universe follows a time-like world line γ through the space-time universe. Suppose that this observer possesses an atomic clock and that he defines a second of time to be a certain number of oscillations of this atomic clock. Then, in traveling along his world line γ from the point $\gamma(a)$ to the point $\gamma(b)$ in the space-time universe, it is postulated that the observer will see that the number of elapsed seconds is given by the proper time integral

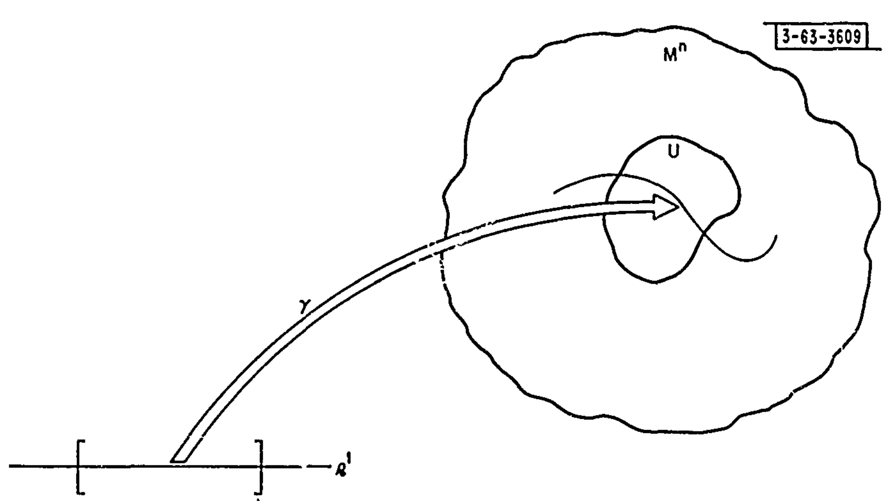

Fig.3. Curve on a manifold.

$$\tau_{ab} = \frac{1}{\bar{c}} \int_a^b \sqrt{g_{\mu\nu} \frac{d\gamma^\mu}{ds} \frac{d\gamma^\nu}{ds}}\, ds \qquad \text{(IV-12)}$$

where $\bar{c}$ is a constant, dependent upon the specific chemical element whose atomic oscillations run the atomic clock and upon the number of oscillations defined to be in a second. The above integral depends only on the world line γ and the gravitational potential $g_{\mu\nu}$. The world line depends on whether the observer is accelerated, etc. Thus, it is only reasonable to postulate (IV-12) for the rate of a clock for an ideal atomic clock, since the effect of impulses on the rate of a mechanical clock would depend on the details of its construction.

Let γ:[a, b] $\rightarrow$ M be a null or time-like curve in the space-time universe M. The length of γ is defined as

$$L(\gamma) = \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}\, ds \quad . \qquad \text{(IV-13)}$$

The curve γ is a geodesic if $L(\gamma)$ is a minimum for all nearby curves joining $\gamma(a)$ and $\gamma(b)$ in M. If the parameter s satisfies $g_{\mu\nu}(dx^\mu/ds)\,(dx^\nu/ds)$ = constant along the curve, then a null or time-like geodesic satisfies the differential equations

$$\frac{d^2x^\beta}{ds^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \qquad \text{(IV-14)}$$

(see Ref. 20). It is postulated that the path followed by a particle of negligible mass through the space-time universe, subject to no force except that due to the gravitational potential in the space-time universe, is a time-like geodesic.

In order to employ the theoretical facade outlined above in concrete situations, we make the following comments. First, in an inertial coordinate system (t, x^1, x^2, x^3) of special relativity far removed from ponderable matter, the gravitational potential should assume the form

$$ds^2 = c^2dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad . \qquad \text{(IV-15)}$$

Here, c is the velocity of light, and dt^2 is short for $dt \otimes dt$. In the spherical coordinate system (t, r, Θ, φ), defined by

$$\begin{aligned} t &= t \\ x^1 &= r \sin\Theta \cos\varphi \\ x^2 &= r \sin\Theta \sin\varphi \\ x^3 &= r \cos\Theta \end{aligned} \qquad \text{(IV-16)}$$

the metric (IV-14) becomes

$$ds^2 = c^2dt^2 - dr^2 - r^2d\Theta^2 - r^2 \sin^2\Theta\, d\varphi^2 \quad . \qquad \text{(IV-17)}$$

Second, in a general relativistic coordinate system which closely approximates a Newtonian inertial coordinate system, the Newtonian expression for the motion in a weak gravitational potential U of a particle of small mass with velocity small relative to the velocity of light should be approximately the same as the general relativistic expression. We therefore have approximately[22]

$$ds^2 = (c^2 + 2U)\, dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad . \tag{IV-18}$$

Here, we assume that U goes to zero at spatial infinity and that the Newtonian acceleration of a particle is −grad U. (The convention in Ref. 22 is that the Newtonian acceleration of a particle is +grad U.) Using the fact that (IV-18) satisfies the Einstein field equations, a more exact expression for the metric is[23]

$$ds^2 = (c^2 + 2U)\, dt^2 - \left(1 - \frac{2U}{c^2}\right)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \quad . \tag{IV-19}$$

B. MOTION OF A PLANET OF SMALL MASS IN THE GRAVITATIONAL FIELD OF THE SUN

Let V be a coordinate neighborhood in the space-time universe containing an isolated spherically symmetric body, and suppose that (t, x^1, x^2, x^3) are coordinates on V such that the center of the body follows the world line given by $x^j = 0$ $(j = 1, 2, 3)$. The gravitational potential on V may be written

$$ds^2 = g_{oo} dt \otimes dt + g_{oi} dt \otimes dx^i + g_{io} dx^i \otimes dt + g_{ij} dx^i \otimes dx^j \quad . \tag{IV-20}$$

Here, and in the following, we assume that Roman indices i, j, ... take on only the values 1, 2, 3. Since the body which generates the gravitational potential (IV-20) is spherically symmetric and isolated, we may suppose that

(1) the gravitational potential is static, i.e., the components of the metric tensor do not depend on the variable t;

(2) the line element (IV-20) does not change its form under a rotation of the coordinate axes (x^1, x^2, x^3);

(3) at a large spatial distance $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ from the body, (IV-20) approaches the value (IV-15).

From these assumptions and the fact that outside the body the Einstein field equations (IV-9) are satisfied, it follows that there is a coordinate system $(t_*, x_*^1, x_*^2, x_*^3)$ on V such that outside the body the metric tensor has components[24]

$$\overset{*}{g}_{oo} = \left(1 - \frac{2\alpha}{r_*}\right) c^2 \quad , \quad \overset{*}{g}_{oj} = 0$$

$$\overset{*}{g}_{ij} = -\delta_{ij} - \frac{2\alpha}{r_* - 2\alpha} \frac{x_*^i x_*^j}{r_*^2} \quad . \tag{IV-21}$$

Here, the Kronecker delta δ_{ij} is defined by (II-60), $r_* = \sqrt{(x_*^1)^2 + (x_*^2)^2 + (x_*^3)^2}$, c is the velocity of light at a large spatial distance from the body, and α is a constant. Comparison of (IV-21) and (IV-18) with $U = -(\gamma M/r_*)$ shows that

$$\alpha = \frac{\gamma M}{c^2} \tag{IV-22}$$

where γ is the gravitational constant, and M is the mass of the body. The constant α has the dimensions of a length and is much smaller than the geometric radius of the body (in the case of the Sun, α = 1.48 km). In the spherical coordinate system $(t_*, r_*, \Theta_*, \varphi_*)$ defined in terms of $(t_*, x_*^1, x_*^2, x_*^3)$ by equations similar to (IV-16), the metric (IV-21) becomes[25]

$$ds^2 = \left(\frac{r_* - 2\alpha}{r_*}\right) c^2 dt_*^2 - \left(\frac{r_*}{r_* - 2\alpha}\right) dr_*^2 - r_*^2 \; (d\theta_*^2 + \sin^2 \theta_* d\varphi_*^2) \quad . \tag{IV-23}$$

The metric given by (IV-21) or (IV-23) is called the Schwarzschild exterior solution of the Einstein field equations. The solution is not valid inside the body, so that the apparent singularity in the metric for $r_* = 2\alpha$ does not really exist.

If we write out (IV-14) for the motion of a body of small mass in the metric (IV-21), we will obtain Newton's equations of motion with a small correction R^k on the right-hand side. We then suppose that this same correction applies to the Newtonian equations of motion of a planet with non-negligible mass acted on by the gravitational attraction of the Sun and other planets and by other forces, obtaining (III-4).. Of course, the rigorously correct procedure would be to derive the equations of motion in a completely relativistic manner, with the equations for the comparison of theory and observation also being derived according to the general theory of relativity. However, given the limitations stated in Sec. I, we continue with the less rigorous procedure of using the relativistic equations of motion of a planet of small mass in the gravitational field of the Sun to correct the Newtonian equations of motion of a planet.

The coordinate system $(t_*, x_*^1, x_*^2, x_*^3)$ could be changed very slightly and the equations of motion of a particle of small mass would still have the appearance of the Newtonian equations of motion with a small (but different) correction on the right-hand side. If we are to follow the plan of correcting the Newtonian equations of motion, this correction should be obtained in the general relativistic coordinate system which most closely fits the Newtonian coordinate system. The only reason that the coordinate system $(t_*, x_*^1, x_*^2, x_*^3)$ with metric (IV-21) could be this "best-fitting" coordinate system is the apparent simplicity of the metric (IV-21).

In a coordinate system (t, x^1, x^2, x^3) with flat metric (IV-15), the d'Alembertian operator $\Box$ is defined on a function f by the equation

$$\Box f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial (x^1)^2} - \frac{\partial^2 f}{\partial (x^2)^2} - \frac{\partial^2 f}{\partial (x^3)^2} \quad . \tag{IV-24}$$

Note that $\Box t = 0$, $\Box x^j = 0$ $(j = 1, 2, 3)$. The natural differential geometric generalization of the d'Alembertian operator to a coordinate system (x^0, x^1, x^2, x^3) with metric $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ is

$$\Box f = g^{\mu\nu} \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} - g^{\mu\nu} \Gamma^\gamma_{\mu\nu} \frac{\partial f}{\partial x^\gamma} \tag{IV-25}$$

where the summation on the Greek indices runs over 0, 1, 2, 3.† A coordinate system (x^0, x^1, x^2, x^3) is harmonic if $\Box x^\mu = 0$ $(\mu = 0, 1, 2, 3)$. Given any coordinate system $(x_*^0, x_*^1, x_*^2, x_*^3)$ and metric, a new coordinate system (x^0, x^1, x^2, x^3) which is harmonic can, in general, be found. (We would have to find four independent solutions of the linear hyperbolic partial differential equation with nonconstant coefficients $\Box f = 0$, which can be done because Cauchy's problem can be solved for this type of partial differential equation.[27]) Suppose we are concerned with an insular distribution of matter contained in a coordinate neighborhood. We may assume that this coordinate neighborhood extends off to infinity, with the insular system being contained in a spatially bounded part of the coordinate neighborhood. Then, Fock[28] proves that if certain natural conditions are satisfied by the metric at spatial infinity, a harmonic coordinate system on the coordinate neighborhood is defined uniquely up to a Lorentz transformation. These conditions essentially state that the metric goes sufficiently fast to the flat space value (III-15) at

†See, for example, Ref. 26.

spatial infinity, and that no gravitational waves impinge on the insular system from the outside.

In the case of the isolated spherically symmetric body considered at the beginning of this section, the harmonic coordinate system (t, x^1, x^2, x^3) can be made unique up to a rotation of the spatial axes by specifying that the world line followed by the center of the spherically symmetric body is given by $x^i = 0$ $(i = 1, 2, 3)$. In these harmonic coordinates, the metric tensor has components[29]

$$g_{oo} = \left(\frac{r-\alpha}{r+\alpha}\right) c^2 \quad , \quad g_{oi} = 0$$

$$g_{ij} = -\left(1 + \frac{\alpha}{r}\right)^2 \delta_{ij} - \left(\frac{r+\alpha}{r-\alpha}\right) \left(\frac{\alpha}{r}\right)^2 \frac{x^i x^j}{r^2} \tag{IV-26}$$

where α is given by (IV-22). In the spherical coordinate system (t, r, Θ, φ), defined in terms of (t, x^1, x^2, x^3) by (IV-16), we have[29]

$$ds^2 = \left(\frac{r-\alpha}{r+\alpha}\right) c^2 dt^2 - \left(\frac{r+\alpha}{r-\alpha}\right) dr^2 - (r+\alpha)^2 \left(d\Theta^2 + \sin^2 \Theta d\varphi^2\right) \quad . \tag{IV-27}$$

The relation between the $(t_*, r_*, \Theta_*, \varphi_*)$ coordinate system of (IV-23) and the (t, r, Θ, φ) coordinate system of (IV-27) is obviously given by

$$t_* = t \quad , \quad r_* = r + \alpha$$

$$\Theta_* = \Theta \quad , \quad \varphi_* = \varphi \tag{IV-28}$$

so that the transformation between the $(t_*, x_*^1, x_*^2, x_*^3)$ coordinate system of (IV-21) and the (t, x^1, x^2, x^3) coordinate system of (IV-26) is given by

$$t_* = t \quad , \quad x_*^i = x^i \left(1 + \frac{\alpha}{r}\right) \quad , \quad i = 1, 2, 3 \quad . \tag{IV-29}$$

The Schwarzschild metric has also been expressed in what are called isotropic coordinates. In isotropic rectangular coordinates $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$, the Schwarzschild metric has components

$$\bar{g}_{oo} = \frac{\left(1 - \frac{\alpha}{2\bar{r}}\right)^2}{\left(1 + \frac{\alpha}{2\bar{r}}\right)^2} c^2 \quad , \quad \bar{g}_{oi} = 0$$

$$\bar{g}_{ij} = -\left(1 + \frac{\alpha}{2\bar{r}}\right)^4 \delta_{ij} \tag{IV-30}$$

while in isotropic spherical coordinates $(\bar{t}, \bar{r}, \bar{\Theta}, \bar{\varphi})$, it is given by

$$ds^2 = \frac{\left(1 - \frac{\alpha}{2\bar{r}}\right)^2}{\left(1 + \frac{\alpha}{2\bar{r}}\right)^2} c^2 dt^2 - \left(1 + \frac{\alpha}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\bar{\Theta}^2 + \bar{r}^2 \sin^2 \bar{\Theta} d\bar{\varphi}^2\right) \tag{IV-31}$$

(see Ref. 30). Comparing (IV-23), (IV-28) and (IV-31), we see that

$$
\left.\begin{aligned}
t_* &= \bar{t} & t &= \bar{t} \\
r_* &= \left(1 + \frac{\alpha}{2\bar{r}}\right)^2 \bar{r} & r &= \left[1 + \left(\frac{\alpha}{2\bar{r}}\right)^2\right]\bar{r} \\
\theta_* &= \bar{\theta} & \theta &= \bar{\theta} \\
\varphi_* &= \bar{\varphi} & \varphi &= \bar{\varphi}
\end{aligned}\right\} \tag{IV-32}
$$

which implies that

$$
\left.\begin{aligned}
t_* &= \bar{t} & t &= \bar{t} \\
x_*^i &= \bar{x}^i \left(1 + \frac{\alpha}{2\bar{r}}\right)^2 & x^i &= \bar{x}^i \left[1 + \left(\frac{\alpha}{2\bar{r}}\right)^2\right] , \quad i = 1, 2, 3
\end{aligned}\right\} . \tag{IV-33}
$$

There are, of course, infinitely many other coordinate systems besides these three in which the Schwarzschild metric can be expressed, but these three are generally used in the literature. We shall now derive the equations of motion of a particle of small mass in each of the coordinate systems discussed above, even though we have reason to believe that the harmonic coordinates are closest to the Newtonian coordinates.

First, we note that, by Ref. 29,

$$
g^{oo} = \frac{1}{c^2}\left(\frac{r+\alpha}{r-\alpha}\right) , \qquad g^{oi} = 0
$$

$$
g^{ij} = \frac{1}{\left(1 + \frac{\alpha}{r}\right)^2}\left[-\delta_{ij} + \left(\frac{\alpha}{r}\right)^2 \frac{x^i x^j}{r^2}\right] \tag{IV-34}
$$

by Ref. 24,

$$
g_*^{oo} = \frac{1}{c^2}\left(\frac{r_*}{r_* - 2\alpha}\right) , \qquad g_*^{oi} = 0
$$

$$
g_*^{ij} = -\delta_{ij} + \frac{2\alpha}{r_*}\frac{x_*^i x_*^j}{r_*^2} \tag{IV-35}
$$

and, by (IV-30),

$$
\bar{g}^{oo} = \frac{1}{c^2}\frac{\left(1 + \frac{\alpha}{2\bar{r}}\right)^2}{\left(1 - \frac{\alpha}{2\bar{r}}\right)^2} , \qquad \bar{g}^{oi} = 0
$$

$$
\bar{g}^{ij} = -\frac{\delta_{ij}}{\left(1 + \frac{\alpha}{2\bar{r}}\right)^4} . \tag{IV-36}
$$

Thus, by definition (IV-4), we have in the (t, x^1, x^2, x^3) coordinate system that

$$\Gamma^{o}_{oo} = 0 \quad , \quad \Gamma^{o}_{ij} = 0 \quad , \quad \Gamma^{k}_{oj} = \Gamma^{k}_{jo} = 0$$

$$\Gamma^{o}_{oj} = \Gamma^{o}_{jo} = \frac{1}{2} g^{oo} \frac{\partial g_{oo}}{\partial x^j} \quad , \quad \Gamma^{k}_{oo} = -\frac{1}{2} \sum_{\ell=1}^{3} g^{k\ell} \frac{\partial g_{oo}}{\partial x^\ell}$$

$$\Gamma^{k}_{ij} = \frac{1}{2} \sum_{\ell=1}^{3} g^{k\ell} \left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right) \tag{IV-37}$$

with exactly similar equations holding in the starred and barred coordinate systems. Equation (IV-14) for a geodesic then becomes

$$\frac{d^2t}{ds^2} + 2 \sum_{j=1}^{3} \Gamma^{o}_{oj} \frac{dt}{ds} \frac{dx^j}{ds} = 0$$

$$\frac{d^2x^k}{ds^2} + \sum_{i,j=1}^{3} \Gamma^{k}_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} + \Gamma^{k}_{oo} \left(\frac{dt}{ds}\right)^2 = 0 \quad . \tag{IV-38}$$

Since

$$\frac{d}{ds} = \frac{dt}{ds} \frac{d}{dt}$$

$$\frac{d^2}{ds^2} = \left(\frac{dt}{ds}\right)^2 \frac{d^2}{dt^2} + \frac{d^2t}{ds^2} \frac{d}{dt}$$

the second equation in (IV-38) can be written

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{3} \Gamma^{k}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \Gamma^{k}_{oo} + \frac{d^2t/ds^2}{(dt/ds)^2} \frac{dx^k}{dt} = 0 \quad .$$

Now, by the first equation in (IV-38), we have

$$\frac{d^2t}{ds^2} = -2\left(\frac{dt}{ds}\right)^2 \sum_{j=1}^{3} \Gamma^{o}_{oj} \frac{dx^j}{dt}$$

so that we can finally write

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{3} \Gamma^{k}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \Gamma^{k}_{oo} - 2 \frac{dx^k}{dt} \sum_{j=1}^{3} \Gamma^{o}_{oj} \frac{dx^j}{dt} = 0 \quad . \tag{IV-39}$$

Exactly similar equations are valid in the starred and barred coordinate systems.

Using (IV-26), (IV-34) and (IV-37), we perform a simple calculation in the (t, x^1, x^2, x^3) coordinate system that gives

$$\Gamma^{o}_{oj} = \frac{\alpha}{(r+\alpha)(r-\alpha)} \frac{x^j}{r}$$

$$\Gamma^{k}_{oo} = \frac{\alpha c^2 x^k}{r^3} \frac{(1-\frac{\alpha}{r})}{(1+\frac{\alpha}{r})^3}$$

$$\Gamma^{k}_{ij} = \frac{\alpha}{r^3}\left\{x^k\delta_{ij} - \frac{1}{1+\frac{\alpha}{r}}\left[x^i\delta_{jk} + x^j\delta_{ik} + \frac{\alpha x^k x^j x^k}{r^3}\left(1+\frac{1}{1-\frac{\alpha}{r}}\right)\right]\right\} \quad . \tag{IV-40}$$

Similarly, using (IV-21), (IV-35) and (IV-37), we find that in the $(t_*, x^1_*, x^2_*, x^3_*)$ coordinate system

$$\Gamma^{*o}_{oj} = \frac{\alpha x^j_*}{(r_* - 2\alpha) r^2_*}$$

$$\Gamma^{*k}_{oo} = \frac{\alpha c^2 x^k_*}{r^3_*}\left(1 - \frac{2\alpha}{r_*}\right)$$

$$\Gamma^{*k}_{ij} = \frac{\alpha x^k_*}{r^3_*}\left[2\delta_{ij} - \frac{x^i_* x^j_*}{r^2_*}\left(2 + \frac{1}{1-\frac{2\alpha}{r_*}}\right)\right] \quad . \tag{IV-41}$$

Finally, using (IV-30), (IV-36) and (IV-37), we find that in the $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ coordinate system

$$\bar{\Gamma}^{o}_{oj} = \frac{\alpha \bar{x}^j}{(1-\frac{\alpha}{2\bar{r}})(1+\frac{\alpha}{2\bar{r}})\bar{r}^3}$$

$$\bar{\Gamma}^{k}_{oo} = \frac{\alpha c^2 \bar{x}^k}{\bar{r}^3} \frac{(1-\frac{\alpha}{2\bar{r}})}{(1+\frac{\alpha}{2\bar{r}})^7}$$

$$\bar{\Gamma}^{k}_{ij} = \frac{\alpha}{(1+\frac{\alpha}{2\bar{r}})\bar{r}^3}(\bar{x}^k\delta_{ij} - \bar{x}^i\delta_{jk} - \bar{x}^j\delta_{ik}) \quad . \tag{IV-42}$$

Formulas (IV-39) and (IV-40) show that the equations of motion of a small mass in the (t, x^1, x^2, x^3) coordinate system are

$$\frac{d^2x^k}{dt^2} + \frac{\gamma M x^k}{r^3} = R^k \tag{IV-43}$$

where

$$R^k = \frac{\gamma M x^k}{r^3}\left[1 - \frac{1-\frac{\alpha}{r}}{(1+\frac{\alpha}{r})^3}\right] - \frac{\gamma M}{c^2 r^3}\left\{x^k \sum_{\ell=1}^{3}\left(\frac{dx^\ell}{dt}\right)^2 - \frac{1}{1+\frac{\alpha}{r}}\left(1+\frac{1}{1+\frac{\alpha}{r}}\right)\left(\sum_{\ell=1}^{3} x^\ell \frac{dx^\ell}{dt}\right)\right.$$

$$\left.\times\left[2\frac{dx^k}{dt} + \frac{\alpha x^k}{r^3}\left(\sum_{\ell=1}^{3} x^\ell \frac{dx^\ell}{dt}\right)\right]\right\} \quad . \tag{IV-44}$$

Similarly, formulas (IV-39) and (IV-41) show that the equations of motion of a particle of small mass in the $(t_*, x_*^1, x_*^2, x_*^3)$ coordinate system are

$$\frac{dx_*^k}{dt_*^2} + \frac{\gamma M x_*^k}{r_*^3} = R_*^k \tag{IV-45}$$

where

$$R_*^k = \frac{\gamma M x_*^k}{r_*^3}\frac{2\alpha}{r_*} - \frac{\gamma M}{c^2 r_*^3}\left\{2x_*^k \sum_{\ell=1}^{3}\left(\frac{dx_*}{dt_*}\right)^2 - \left(\sum_{\ell=1}^{3} x_*^\ell \frac{dx_*^\ell}{dt_*}\right)\left[\frac{2}{1-\frac{2\alpha}{r_*}}\frac{dx_*^k}{dt_*} + \left(2 + \frac{1}{1-\frac{2\alpha}{r_*}}\right)\left(\sum_{\ell=1}^{3} x_*^\ell \frac{dx_*^\ell}{dt_*}\right)\frac{x_*^k}{r_*^2}\right]\right\} \quad . \tag{IV-46}$$

Finally, formulas (IV-39) and (IV-42) show that the equations of motion of a particle of small mass in the $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ coordinate system are

$$\frac{d\bar{x}^k}{d\bar{t}^2} + \frac{\gamma M \bar{x}^k}{\bar{r}^3} = \bar{R}^k \tag{IV-47}$$

where

$$\bar{R}^k = \frac{\gamma M \bar{x}^k}{\bar{r}^3}\left[1 + \frac{\left(1-\frac{\alpha}{2\bar{r}}\right)}{\left(1+\frac{\alpha}{2\bar{r}}\right)^7}\right] - \frac{\gamma M}{c^2 \bar{r}^3}\frac{1}{\left(1+\frac{\alpha}{2\bar{r}}\right)}\left[\bar{x}^k \sum_{\ell=1}^{3}\left(\frac{d\bar{x}^\ell}{d\bar{t}}\right)^2 - 2\left(1 + \frac{1}{\left(1-\frac{\alpha}{2\bar{r}}\right)}\right)\left(\sum_{\ell=1}^{3} \bar{x}^\ell \frac{d\bar{x}^\ell}{d\bar{t}}\right)\frac{d\bar{x}^k}{d\bar{t}}\right] \tag{IV-48}$$

In the case of Mercury,

$$\frac{\alpha}{r} \approx \frac{\alpha}{r_*} \approx \frac{\alpha}{\bar{r}} \approx 3 \times 10^{-8}$$

$$\frac{v}{c} \approx \frac{v_*}{c} \approx \frac{\bar{v}}{c} \approx 1.5 \times 10^{-4} \quad .$$

Thus, using the fact that for small z

$$\frac{1}{(1 \pm z)^n} \approx 1 \mp nz$$

we drop all terms in (IV-44), (IV-46) and (IV-48) which contain $(\alpha/r)^2$ or $(\alpha/r)\,(v/c)^2$ as factors, obtaining,

$$R^k = \frac{\gamma M x^k}{r^3}\left[\frac{4\alpha}{r} - \frac{1}{c^2}\sum_{\ell=1}^{3}\left(\frac{dx^\ell}{dt}\right)^2\right] + \frac{4\gamma M}{c^2 r^3}\frac{dx^k}{dt}\left(\sum_{\ell=1}^{3} x^\ell \frac{dx^\ell}{dt}\right) \tag{IV-49}$$

$$R_*^k = \frac{\gamma M x_*^k}{r_*^3}\left[\frac{2\alpha}{r_*} - \frac{2}{c^2}\sum_{\ell=1}^{3}\left(\frac{dx_*^\ell}{dt_*}\right)^2 + \frac{3}{c^2 r_*^2}\left(\sum_{\ell=1}^{3} x_*^\ell \frac{dx_*^\ell}{dt_*}\right)^2\right] + \frac{2\gamma M}{c^2 r_*^3}\frac{dx_*^k}{dt_*}\left(\sum_{\ell=1}^{3} x_*^\ell \frac{dx_*^\ell}{dt_*}\right) \tag{IV-50}$$

$$\bar{R}^k = \frac{\gamma M \bar{x}^k}{\bar{r}^3}\left[\frac{4\alpha}{\bar{r}} - \frac{1}{c^2}\sum_{\ell=1}^{3}\left(\frac{d\bar{x}^\ell}{d\bar{t}}\right)^2\right] + \frac{4\gamma M}{c^2 \bar{r}^3}\frac{d\bar{x}^k}{d\bar{t}}\sum_{\ell=1}^{3}\bar{x}^\ell \frac{d\bar{x}^\ell}{d\bar{t}} \quad . \tag{IV-51}$$

Equations (IV-43), (IV-45) and (IV-47) are invariant under rotation of the coordinate axes. From this it easily follows that the motion given by these equations lies in a plane in each of the three coordinate systems.

By coordinate transformations (IV-28), (IV-29), (IV-32) and (IV-33), the curves in the three coordinate systems given by (IV-43), (IV-45) and (IV-47) are exactly similar, even though these equations have dissimilar appearance. As is well known, these curves are ellipses with advancing perihelions.[31, 32, 33] The expressions for the periods of these ellipses, in terms of the semi-major axes of the ellipses, will vary in the different coordinate systems because of relations (IV-28) and (IV-32).

We have two candidates for the relativity term in (III-4) and (III-12). Because of the harmonic and isotropic criteria, and for the sake of definitiveness, we choose (IV-49) [or equivalently (IV-51)] to be this term. In the notation of Sec. III-A, it is

$$R^k = \frac{\gamma M_s R_f}{r_{ps}^3}\left\{x_{ps}^k\left[4\frac{\alpha}{r_{ps}} - \frac{1}{c^2}\sum_{\ell=1}^{3}\left(\frac{dx^\ell}{dt}\right)^2\right] + \frac{4}{c^2}\frac{dx_{ps}^k}{dt}\left(\sum_{\ell=1}^{3} x_{ps}^\ell \frac{dx_{ps}^\ell}{dt}\right)\right\} \quad , \quad k = 1, 2, 3 \tag{IV-52}$$

where we have multiplied (IV-49) by a dimensionless constant R_f. If we perform a least-squares analysis on the value of R_f and other parameters to fit theory and observation, R_f will converge to the value 1 if the relativity correction belongs in the equations of motion, or to the value 0 if the Newtonian theory is correct. By (IV-22),

$$\alpha = \frac{\gamma M_s}{c^2} \tag{IV-53}$$

where c is the velocity of light at a large spatial distance from the Sun.

C. METHOD OF SOLVING THE PROBLEM OF THE MOTION OF A SYSTEM OF MASSES IN GENERAL RELATIVITY

If we raise indices in the Einstein field equations (IV-7), we obtain

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\kappa T^{\mu\nu} \quad . \tag{IV-5?}$$

These equations are nonlinear and hyperbolic in the unknown functions $g^{\mu\nu}$. The fact that they are hyperbolic implies that gravitational waves can exist. Their nonlinearity allows us not only to determine the potential $g^{\mu\nu}$, but also the mass tensor $T^{\mu\nu}$, i.e., the motion of the masses. In all other field theories, such as Maxwell's for the electromagnetic field or Newton's for the gravitational field, the field equations are linear and the equations for the motion of bodies in the field are separate from and additional to the field equations. But in Einstein's theory, the equations of motion are contained in the equations for the field.

The divergence of the left side of the Einstein field equations vanishes,

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0 \quad . \tag{IV-55}$$

(This was one of the attributes which led Einstein to choose this tensor for the left side of his equation.) Thus, we have

$$\nabla_\mu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + T^{\sigma\nu}\Gamma^{\mu}_{\sigma\mu} + T^{\mu\sigma}\Gamma^{\nu}_{\sigma\mu} = 0 \quad . \tag{IV-56}$$

The simultaneous solution of (IV-54) and (IV-56) will determine the field and the motion of the masses; of course, (IV-56) is a consequence of (IV-54).

To obtain an approximate expression for the mass tensor, let $\rho(x^0, x^1, x^2, x^3)$ be (in some sense) the invariant density of matter in the space-time universe. We suppose that

$$T^{\mu\nu} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \tag{IV-57}$$

where s is the proper time of the element of matter at the point (x^0, x^1, x^2, x^3). If the element of matter is following a world line $x^\mu = x^\mu(s)$, then the defining property of s is

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1 \quad . \tag{IV-58}$$

If we imagine that we are concerned with a particle of small mass and small dimensions which has negligible effect on the gravitational field, (IV-56) and (IV-57) imply that the particle follows a time-like geodesic through the space-time universe.[34] Thus, assumption (IV-14) is not really an assumption, but is a consequence of the Einstein field equations.

The method of solving the field equations for the field and the motion of the masses is presented by Fock,[21] and by Infeld and Plebanski.[35] The latter follow the work of Einstein, Infeld and Hoffmann[36] and assume that the masses are point singularities of the field, so that the mass tensor is zero everywhere except along the world lines of the particles, where it is given by delta functions. Fock assumes a continuous distribution of matter concentrated in a finite number of regions, so that the mass tensor is differentiable everywhere, and is zero outside of the finite number of regions. The methods used by Fock and by Infeld and Plebanski are approximation procedures and are essentially the same; Fock assumes that he is always working in a harmonic coordinate system, while Infeld and Plebanski make supplementary coordinate conditions at each step in the approximation.

To be specific, let us outline Fock's procedure with a continuous distribution of matter concentrated in a finite number of regions. We first assume expression (IV-57) for the mass tensor $T^{\mu\nu}$ (at a later stage in the approximation, we can assume a more sophisticated form of the mass tensor using the fact that the bodies are elastic). Then we solve (IV-54) for the gravitational potential $g^{\mu\nu}$ to first order in v/c, obtaining (IV-19) with some additional terms of the form $dt \otimes dx^i$ times quantities involving the velocity of the matter generating the field. This is called the Newtonian approximation. We use this solution for the potential to write equations (IV-56) for the mass tensor $T^{\mu\nu}$ to first order in v/c. The solution of these equations is used to solve (IV-54) for the gravitational potential $g^{\mu\nu}$ to second order in v/c. This solution for the potential is then used to write equations (IV-56) for the mass tensor $T^{\mu\nu}$ to second order in v/c. We could, in principle, continue this procedure indefinitely, but the solutions and equations in this post-Newtonian approximation are accurate enough for our purposes.

At whatever stage we stop in the approximation procedure, we will have found in a specific coordinate system a system of second order ordinary differential equations for the motions of the masses, and an expression for the gravitational potential in terms of the motions of masses. If the coordinate system is (t, x^1, x^2, x^3), t can be made the independent variable of the equations of motion. Numerical integration of the equations of motion will determine ephemerides in this specific coordinate system of the planets as functions of t. The relation between the coordinate time t and the proper time τ of an atomic clock on the surface of the Earth following a world line $\gamma(s)$ $(s = t)$ is given by (IV-12). This integral can be evaluated because we know the gravitational potential in our specific coordinate system. Knowing the general relativistic theory of radar and optical observations of a planet, we can compute the theoretical values of observations made at given instants of atomic time. Then, making a least-squares adjustment to the initial conditions and parameters appearing in the theory of motion, we can determine the general relativistic ephemerides which best fit observation.

This report is concerned with Newtonian theory and any general relativistic corrections that are easily obtained. The procedure outlined above can and should be documented and pursued; this we hope to do, following Fock[21] and Infeld and Plebanski.[35]

V. EQUATIONS FOR PARTIAL DERIVATIVES OF POSITION AND VELOCITY WITH RESPECT TO INITIAL CONDITIONS AND PARAMETERS

A. PLANET CASE

Let $(x_{ps}^1, \ldots, x_{ps}^6)$ denote the components of position and velocity of a planet relative to the Sun. Equation (III-4) for the motion of the planet can be written in the form

$$\left.\begin{aligned}
\frac{dx_{ps}^k}{dt} &= x_{ps}^{k+3} \\
\frac{dx_{ps}^{k+3}}{dt} &= -\gamma M_s \left(1 + \frac{M_p}{M_s}\right) \frac{x_{ps}^k}{r_{ps}^3} + \Omega^k + R^k + S^k + \frac{1}{M_p} F_p^k \\
x_{ps}^k &= x_{ops}^k \quad , \quad x_{ps}^{k+3} = x_{ops}^{k+3} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-1}$$

Let α be a parameter upon which the motion of the planet depends, such as an initial condition, a planetary mass, the second harmonic of the Sun, etc. Differentiating system (V-1) with respect to α, we see that the quantities $(\partial x_{ps}^1/\partial\alpha, \ldots, \partial x_{ps}^6/\partial\alpha)$ satisfy the differential equations system

$$\left.\begin{aligned}
\frac{d(\partial x_{ps}^k/\partial\alpha)}{dt} &= \frac{\partial x_{ps}^{k+3}}{\partial\alpha} \\
\frac{d(\partial x_{ps}^{k+3}/\partial\alpha)}{dt} &= \gamma M_s \left(1 + \frac{M_p}{M_s}\right) \frac{1}{r_{ps}^3} \left(\frac{3x_{ps}^k}{r_{ps}^2} \sum_{\ell=1}^{3} x_{ps}^\ell \frac{\partial x_{ps}^\ell}{\partial\alpha} - \frac{\partial x_{ps}^k}{\partial\alpha}\right) \\
&\quad - \left(1 + \frac{M_p}{M_s}\right) \frac{x_{ps}^k}{r_{ps}^3} \frac{\partial(\gamma M_s)}{\partial\alpha} - \gamma M_s \frac{x_{ps}^k}{r_{ps}^3} \frac{\partial(M_p/M_s)}{\partial\alpha} \\
&\quad + \frac{\partial\Omega^k}{\partial\alpha} + \frac{\partial R^k}{\partial\alpha} + \frac{\partial S^k}{\partial\alpha} + \frac{\partial}{\partial\alpha}\left(\frac{1}{M_p} F_p^k\right) \\
\frac{\partial x_{ps}^k}{\partial\alpha} &= \frac{\partial x_{ops}^k}{\partial\alpha} \quad , \quad \frac{\partial x_{ops}^{k+3}}{\partial\alpha} = \frac{\partial x_{ops}^{k+3}}{\partial\alpha} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-2}$$

We have $\partial x_{ops}^\ell/\partial\alpha = 0$ ($\ell = 1, \ldots, 6$) for all parameters α which are not initial conditions. The value of $\partial x_{ops}^\ell/\partial\alpha$ for an initial condition α depends on the specific initial condition. For example, if we let $\alpha = x_{ops}^j$, then $\partial x_{ops}^\ell/\partial\alpha = \delta_{\ell j}$. We shall choose the initial conditions as $(\beta_p^1, \ldots, \beta_p^6) = (a, e, i, \Omega, \omega, \ell_o)$, the orbital elements of the elliptic orbit osculating to the true orbit of the planet at the initial time t_o, because we will be integrating differential equations system (V-2) for $\partial x_{ps}^k/\partial\beta_p^j$ (j, k = 1, ..., 6) by means of Encke's method as explained in Sec. VI-A. Also, we are going to use the results of the integration to make a least-squares correction to the initial conditions, and it is more meaningful physically to adjust $(\beta_p^1, \ldots, \beta_p^6)$ than to adjust $(x_{ops}^1, \ldots, x_{ops}^6)$. The relation between $(\beta_p^1, \ldots, \beta_p^6)$ and $(x_{ops}^1, \ldots, x_{ops}^6)$ is given in Secs. II-B and II-C, while the values of $\partial x_{ops}^k/\partial\beta_p^j$ are given in formulas (II-28) to (II-33) with $t = t_o$.

If we desire to take account of a possible time variation of the gravitational constant, we might suppose that

$$\gamma M_s = (\gamma M_s)_o \, [1 + \lambda(t - t_o)] \tag{V-3}$$

where λ is a parameter to be determined by comparing theory with observation. We then have

$$\frac{\partial(\gamma M_s)}{\partial\lambda} = (\gamma M_s)_o \, (t - t_o) \tag{V-4}$$

with the $\partial(\gamma M_s)/\partial\alpha$ term in (V-2) being zero for all other parameters α.†

The term $\partial(M_p/M_s)/\partial\alpha$ in (V-2) is zero for all parameters α, except for $\alpha = M_p/M_s$, in which case it takes the value 1.

To determine the term $\partial\Omega^k/\partial\alpha$ in (V-2), we differentiate (III-3) with respect to α, obtaining

$$\frac{\partial\Omega^k}{\partial\alpha} = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \frac{1}{r_{jp}^3} \left(\frac{3x_{jp}^k}{r_{jp}^2} \sum_{\ell=1}^{3} x_{jp}^{\ell} \frac{\partial x_{ps}^{\ell}}{\partial\alpha} - \frac{\partial x_{ps}^k}{\partial\alpha} \right) \quad , \quad k = 1, 2, 3 \tag{V-5}$$

if α is not the mass of a perturbing planet. Here, we have to assume that

$$\left. \begin{aligned} &\frac{\partial x_{js}^k}{\partial\alpha} = 0 \\ &\frac{\partial x_{jp}^k}{\partial\alpha} = \frac{\partial x_{js}^k}{\partial\alpha} - \frac{\partial x_{ps}^k}{\partial\alpha} = -\frac{\partial x_{ps}^k}{\partial\alpha} \end{aligned} \right\} \quad k = 1, 2, 3 \tag{V-6}$$

since it is supposed that $(x_{js}^1, x_{js}^2, x_{js}^3)$ are given as definite functions of time. If α is the mass of some perturbing planet, $\alpha = M_i/M_s$, then we must add the following term to (V-5):

$$\gamma M_s \left(\frac{x_{ip}^k}{r_{ip}^3} - \frac{x_{is}^k}{r_{is}^3} \right) \quad , \quad k = 1, 2, 3 \quad . \tag{V-7}$$

If $\alpha = \lambda$, we must add this term to (V-5):

$$(\gamma M_s)_o \, (t - t_o) \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{x_{jp}^k}{r_{jp}^3} - \frac{x_{js}^k}{r_{js}^3} \right) \quad , \quad k = 1, 2, 3 \tag{V-8}$$

If α is an initial condition for planet p, assumption (V-6) that $\partial x_{js}^k/\partial\alpha$ is zero relative to $\partial x_{ps}^k/\partial\alpha$ is certainly justified. However, if α is some parameter other than an initial condition, $\partial x_{js}^k/\partial\alpha$ could be comparable or even larger than $\partial x_{ps}^k/\partial\alpha$ so that expression (V-5) for $\partial\Omega^k/\partial\alpha$ would be incorrect in nature. For use in PEP, (V-5) is exactly correct when the perturbing planet input magnetic tape used in evaluating (III-3) is kept the same between iterations, but not when the ephemerides on the perturbing planet tape are replaced by the results of just completed

† We do not consider $(\gamma M_s)_o$ as a parameter to be adjusted because it is the usual practice in celestial mechanics to set $\sqrt{(\gamma M_s)_o} = 0.01720209895$, which defines the unit of length (the Astronomical Unit) once the unit of time has been specified.

integrations for use in the next iteration. In any case, because (V-5) contains the factor (M_j/M_s), it is less important than (V-7) or the first term on the right of the second equation in (V-2). Since we only need approximate values for the partial derivatives in the iterative process of finding least-squares corrections to the various parameters, assumption (V-6) is thus reasonable. From an operational standpoint, it is necessary.

It is probably sufficiently accurate to suppose that $\partial R^k/\partial\alpha = 0$ and $\partial S^k/\partial\alpha = 0$, except when $\alpha = R_f$ or $\alpha = S_2/R_s^2$, respectively. However, for the sake of completeness, we shall evaluate these quantities. First, differentiating (IV-52) with respect to $\bar{\alpha}$, we see that

$$\frac{\partial R^k}{\partial\bar{\alpha}} = \frac{(\gamma M_s) R_f}{r_{ps}^3}\left\{\frac{\partial x_{ps}^k}{\partial\bar{\alpha}}\left(4\frac{\alpha}{r_{ps}} - \frac{v_{ps}^2}{c^2}\right) + \frac{4}{c^2}(\vec{r}_{ps}\cdot\vec{v}_{ps})\frac{\partial x_{ps}^{k+3}}{\partial\bar{\alpha}}\right.$$

$$- x_{ps}^k\left[\left(\sum_{\ell=1}^{3} x_{ps}^\ell \frac{\partial x_{ps}^\ell}{\partial\bar{\alpha}}\right)\left(\frac{16\alpha}{r_{ps}^3} - \frac{3v_{ps}^2}{c^2 r_{ps}^2}\right) + \frac{2}{c^2}\sum_{\ell=4}^{6} x_{ps}^\ell \frac{\partial x_{ps}^\ell}{\partial\bar{\alpha}}\right]$$

$$- \frac{4x_{ps}^{k+3}}{c^2}\left[\frac{3(\vec{r}_{ps}\cdot\vec{v}_{ps})}{r_{ps}^2}\sum_{\ell=1}^{3} x_{ps}^\ell \frac{\partial x_{ps}^\ell}{\partial\bar{\alpha}} - \sum_{\ell=1}^{3}\left(x_{ps}^\ell \frac{\partial x_{ps}^{\ell+3}}{\partial\bar{\alpha}}\right.\right.$$

$$\left.\left.\left. + \frac{\partial x_{ps}^\ell}{\partial\bar{\alpha}} x_{ps}^{\ell+3}\right)\right]\right\} \quad , \quad k = 1, 2, 3 \qquad \text{(V-9)}$$

where

$$v_{ps}^2 = \sum_{\ell=4}^{6} (x_{ps}^\ell)^2$$

$$\vec{r}_{ps}\cdot\vec{v}_{ps} = \sum_{\ell=1}^{3} x_{ps}^\ell x_{ps}^{\ell+3} \quad . \qquad \text{(V-10)}$$

We have denoted the parameter with which we differentiate (IV-52) by $\bar{\alpha}$ so that there will be no confusion with the gravitational radius of the Sun α appearing in this formula. If $\bar{\alpha} = R_f$, we must add the following term to (V-9):

$$\frac{\gamma M_s}{r_{ps}^3}\left[x_{ps}^k\left(4\frac{\alpha}{r_{ps}} - \frac{1}{c^2}v_{ps}^2\right) + \frac{4x_{ps}^{k+3}}{c^2}(\vec{r}_{ps}\cdot\vec{v}_{ps})\right] \quad , \quad k = 1, 2, 3 \quad . \qquad \text{(V-11)}$$

If $\bar{\alpha} = \lambda$, we must add this term to (V-9):

$$\frac{(\gamma M_s)_o (t - t_o) R_f}{r_{ps}^3}\left[x_{ps}^k\left(4\frac{\alpha}{r_{ps}} - \frac{1}{c^2}v_{ps}^2\right) + \frac{4x_{ps}^{k+3}}{c^2}(\vec{r}_{ps}\cdot\vec{v}_{ps})\right] \quad , \quad k = 1, 2, 3 \quad . \qquad \text{(V-12)}$$

Second, differentiating (III-50) with respect to α, we see that

$$\frac{\partial S^k}{\partial \alpha} = \frac{\gamma M_s\left(1+\frac{M_p}{M_s}\right)}{r_{ps}^2}\left(\frac{R_s}{r_{ps}}\right)^2 \frac{S_2}{R_s^2}\left[\frac{1}{r_{ps}}\left(\frac{\partial x_{ps}^k}{\partial \alpha} - \frac{5x_{ps}^k}{r_{ps}^2}\sum_{\ell=1}^{3} x_{ps}^{\ell}\frac{\partial x_{ps}^{\ell}}{\partial \alpha}\right)\right.$$

$$\left.\times\left(\frac{15}{2}g^2 - \frac{3}{2}\right) + \frac{\partial g}{\partial \alpha}\left(15g\frac{x_{ps}^k}{r_{ps}} - 3C_{3k}\right)\right] \quad , \quad k = 1, 2, 3 \tag{V-13}$$

where, by (III-51),

$$\frac{\partial g}{\partial \alpha} = \frac{1}{r_{ps}}\sum_{\ell=1}^{3} C_{3\ell}\left(\frac{\partial x_{ps}^{\ell}}{\partial \alpha} - \frac{x_{ps}^{\ell}}{r_{ps}^2}\sum_{i=1}^{3} x_{ps}^{i}\frac{\partial x_{ps}^{i}}{\partial \alpha}\right) \quad . \tag{V-14}$$

If $\alpha = S_2/R_s^2$, we must add the following term to (V-13):

$$\frac{\gamma M_s\left(1+\frac{M_p}{M_s}\right)}{r_{ps}^2}\left(\frac{R_s}{r_{ps}}\right)^2\left[\frac{x_{ps}^k}{r_{ps}}\left(\frac{15}{2}g^2 - \frac{3}{2}\right) - 3gC_{3k}\right] \quad , \quad k = 1, 2, 3 \quad . \tag{V-15}$$

If $\alpha = M_p/M_s$, we must add this term to (V-13):

$$\frac{\gamma M_s}{r_{ps}^2}\left(\frac{R_s}{r_{ps}}\right)^2 \frac{S_2}{R_s^2}\left[\frac{x_{ps}^k}{r_{ps}}\left(\frac{15}{2}g^2 - \frac{3}{2}\right) - 3gC_{3k}\right] \quad , \quad k = 1, 2, 3 \quad . \tag{V-16}$$

If $\alpha = \lambda$, we must add this term to (V-13):

$$\frac{(\gamma M_s)_o\,(t - t_o)\left(1+\frac{M_p}{M_s}\right)}{r_{ps}^2}\left(\frac{R_s}{r_{ps}}\right)^2 \frac{S_2}{R_s^2}\left[\frac{x_{ps}^k}{r_{ps}}\left(\frac{15}{2}g^2 - \frac{3}{2}\right) - 3gC_{3k}\right] \quad , \quad k = 1, 2, 3 \quad . \tag{V-17}$$

B. EARTH-MOON CASE

Let $(x_{cs}^1, \ldots, x_{cs}^6)$ and $(x_{me}^1, \ldots, x_{me}^6)$ denote the components of position and velocity of the Earth-Moon barycenter relative to the Sun, and of the Moon relative to the Earth, respectively. These components will be considered as primary quantities determined by integrating the equations of motion. The components of position and velocity of the Earth and Moon relative to the Sun $(x_{es}^1, \ldots, x_{es}^6)$ and $(x_{ms}^1, \ldots, x_{ms}^6)$ are determined from them by formulas (III-6). Equations (III-12) for the motions of the Earth-Moon barycenter and the Moon can be written in the form

$$\left.\begin{aligned}
&\frac{dx_{cs}^k}{dt} = x_{cs}^{k+3} \\
&\frac{dx_{cs}^{k+3}}{dt} = -\gamma M_s\left(1+\frac{M_c}{M_s}\right)\left(\frac{M_e}{M_c}\frac{x_{es}^k}{r_{es}^3} + \frac{M_m}{M_c}\frac{x_{ms}^k}{r_{ms}^3}\right) \\
&\qquad + \Phi^k + R^k + S^k + \frac{1}{M_c}\left(F_e^k + F_m^k\right) \\
&x_{cs}^k = x_{ocs}^k \quad , \quad x_{cs}^{k+3} = x_{ocs}^{k+3} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \tag{V-18}$$

$$\left.\begin{aligned}
&\frac{dx_{me}^{k}}{dt} = x_{me}^{k+3} \\
&\frac{dx_{me}^{k+3}}{dt} = -\gamma M_s \frac{M_c}{M_s} \frac{x_{me}^{k}}{r_{me}^{3}} + B^k + \Psi^k + H^k + \left(\frac{1}{M_m} F_m^k - \frac{1}{M_e} F_e^k\right) \\
&x_{me}^{k} = x_{ome}^{k} \quad , \quad x_{me}^{k+3} = x_{ome}^{k+3} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \ . \qquad \text{(V-19)}$$

Let α be some parameter upon which the motions of the Earth-Moon barycenter and of the Moon depend. Differentiating systems (V-18) and (V-19) with respect to α, we see that the quantities $(\partial x_{cs}^{1}/\partial\alpha, \ldots, \partial x_{cs}^{6}/\partial\alpha)$ and $(\partial x_{me}^{1}/\partial\alpha, \ldots, \partial x_{me}^{6}/\partial\alpha)$ satisfy the differential equations systems

$$\left.\begin{aligned}
&\frac{d(\partial x_{cs}^{k}/\partial\alpha)}{dt} = \frac{\partial x_{cs}^{k+3}}{\partial\alpha} \\
&\frac{d(\partial x_{cs}^{k+3}/\partial\alpha)}{dt} = -\gamma M_s \left(1 + \frac{M_c}{M_s}\right) \frac{\partial}{\partial\alpha}\left(\frac{M_e}{M_c}\frac{x_{es}^{k}}{r_{es}^{3}} + \frac{M_m}{M_c}\frac{x_{ms}^{k}}{r_{ms}^{3}}\right) \\
&\qquad -\left(\frac{M_e}{M_c}\frac{x_{es}^{k}}{r_{es}^{3}} + \frac{M_m}{M_c}\frac{x_{ms}^{k}}{r_{ms}^{3}}\right)\left[\left(1 + \frac{M_c}{M_s}\right)\frac{\partial(\gamma M_s)}{\partial\alpha}\right. \\
&\qquad \left. + \gamma M_s \frac{\partial(M_c/M_s)}{\partial\alpha}\right] + \frac{\partial\Phi^k}{\partial\alpha} + \frac{\partial R^k}{\partial\alpha} + \frac{\partial S^k}{\partial\alpha} \\
&\qquad + \frac{\partial}{\partial\alpha}\left[\frac{1}{M_c}(F_e^k + F_m^k)\right] \\
&\frac{\partial x_{cs}^{k}}{\partial\alpha} = \frac{\partial x_{ocs}^{k}}{\partial\alpha} \quad , \quad \frac{\partial x_{cs}^{k+3}}{\partial\alpha} = \frac{\partial x_{ocs}^{k+3}}{\partial\alpha} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \qquad \text{(V-20)}$$

$$\left.\begin{aligned}
&\frac{d(\partial x_{me}^{k}/\partial\alpha)}{dt} = \frac{\partial x_{me}^{k+3}}{\partial\alpha} \\
&\frac{d(\partial x_{me}^{k+3}/\partial\alpha)}{dt} = \gamma M_s \frac{M_c}{M_s}\frac{1}{r_{me}^{3}}\left(\frac{3x_{me}^{k}}{r_{me}^{2}}\sum_{\ell=1}^{3} x_{me}^{\ell}\frac{\partial x_{me}^{\ell}}{\partial\alpha} - \frac{\partial x_{me}^{k}}{\partial\alpha}\right) \\
&\qquad - \frac{M_c}{M_s}\frac{x_{me}^{k}}{r_{me}^{3}}\frac{\partial\gamma M_s}{\partial\alpha} - \gamma M_s \frac{x_{me}^{k}}{r_{me}^{3}}\frac{\partial(M_c/M_s)}{\partial\alpha} + \frac{\partial B^k}{\partial\alpha} \\
&\qquad + \frac{\partial\Psi^k}{\partial\alpha} + \frac{\partial H^k}{\partial\alpha} + \frac{\partial}{\partial\alpha}\left(\frac{1}{M_m}F_m^k - \frac{1}{M_e}F_e^k\right) \\
&\frac{\partial x_{me}^{k}}{\partial\alpha} = \frac{\partial x_{ome}^{k}}{\partial\alpha} \quad , \quad \frac{\partial x_{me}^{k+3}}{\partial\alpha} = \frac{\partial x_{ome}^{k+3}}{\partial\alpha} \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \ . \qquad \text{(V-21)}$$

Let $(\beta_c^1, \ldots, \beta_c^6)$ and $(\beta_m^1, \ldots, \beta_m^6)$ be the orbital elements of the elliptic orbits osculating at the initial time to the true orbits of the Earth-Moon barycenter around the Sun and of the Moon around the Earth, respectively. Since we shall be integrating the differential equations for the partial derivatives with respect to initial conditions by means of Encke's method, we choose the initial conditions with respect to which we take partial derivatives to be $(\beta_c^1, \ldots, \beta_c^6)$ and $(\beta_m^1, \ldots, \beta_m^6)$. The initial conditions $\partial x_{ocs}^k/\partial\beta_c^j$ and $\partial x_{ome}^k/\partial\beta_m^j$ $(j, k = 1, \ldots, 6)$ are then determined by the elliptic orbit formulas of Sec. II-D. Of course, we have

$$\left.\begin{aligned} \frac{\partial x_{ocs}^k}{\partial\beta_m^j} &= 0 \\ \frac{\partial x_{ome}^k}{\partial\beta_c^j} &= 0 \end{aligned}\right\} \quad j, k = 1, \ldots, 6 \quad . \tag{V-22}$$

Further, the initial conditions of (V-20) and (V-21) are zero for any parameter α not an initial condition.

We divide the initial conditions and parameters appearing in the theories of motion of the Earth-Moon barycenter and of the Moon into the following three classes:

$$\left.\begin{array}{ll} \text{Initial osculating orbital elements of orbit of Earth-Moon barycenter about the Sun} & (\beta_c^1, \ldots, \beta_c^6) \\ \dfrac{\text{Mass of } j^{th} \text{ perturbing planet}}{\text{Mass of Sun}} & M_{js} = \dfrac{M_j}{M_s} \\ \text{Relativity factor} & R_f \\ \text{Second harmonic of the Sun} & \bar{S}_2 = \dfrac{S_2}{R_s} \end{array}\right\} \quad . \tag{V-23}$$

$$\begin{array}{ll} \text{Initial osculating orbital elements of orbit of the Moon about the Earth} & (\beta_m^1, \ldots, \beta_m^6) \end{array} \quad . \tag{V-24}$$

$$\left.\begin{array}{ll} \dfrac{\text{Mass of Earth + Moon}}{\text{Mass of Sun}} & M_{cs} = \dfrac{M_c}{M_s} \\ \dfrac{\text{Mass of Moon}}{\text{Mass of Earth + Moon}} & M_{mc} = \dfrac{M_m}{M_c} \\ \text{Time variation factor for gravitational constant} & \lambda \end{array}\right\} \quad . \tag{V-25}$$

For parameters α of the form (V-23), we shall assume that $\partial x_{me}^k/\partial\alpha = 0$, and for (V-24) that $\partial x_{cs}^k/\partial\alpha = 0$. We make no such assumptions concerning parameters (V-25). Comparing the magnitude of the A^k term in (III-13) given by (III-32) with the magnitude of the perturbing planet effects in Table II, we see that it is indeed reasonable to assume that $\partial x_{cs}^k/\partial\beta_m^j = 0$, since we must assume that Φ^k is independent of the initial conditions for the perturbing planets. The assumption that $\partial x_{me}^k/\partial\beta_c^j = 0$ is not precisely true, but according to Table III these derivatives are much smaller than the $\partial x_{me}^k/\partial\beta_m^j$ and we have to make some such assumption to make our problem manageable. Comparing Tables II and III, we see that it is reasonable to assume that

the $\partial x_{me}^k/\partial M_{js}$ are zero relative to the $\partial x_{cs}^k/\partial M_{js}$. (We repeat that in the iterative determination of the least-squares correction to the various parameters, only the approximately exact values of the partial derivatives with respect to these parameters are needed, not the exact values.)

For parameters of the type (V-23) for which we assume $\partial x_{me}^k/\partial\alpha = 0$, the equations in (III-6) give

$$\left.\begin{aligned} \frac{\partial x_{es}^k}{\partial\alpha} &= \frac{\partial x_{cs}^k}{\partial\alpha} \\ \frac{\partial x_{ms}^k}{\partial\alpha} &= \frac{\partial x_{cs}^k}{\partial\alpha} \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-26}$$

We now determine the quantities on the right side of (V-20) for these parameters α. First, of course, $\partial(\gamma M_s)/\partial\alpha = 0$ and $(\partial/\partial\alpha)\,(M_c/M_s) = 0$. Next, we have

$$\begin{aligned} \frac{\partial}{\partial\alpha}\left(\frac{M_e}{M_c}\frac{x_{es}^k}{r_{es}^3} + \frac{M_m}{M_c}\frac{x_{ms}^k}{r_{ms}^3}\right) &= \frac{\partial x_{cs}^k}{\partial\alpha}\left(\frac{M_e}{M_c}\frac{1}{r_{es}^3} + \frac{M_m}{M_c}\frac{1}{r_{ms}^3}\right) - 3\sum_{\ell=1}^{3}\frac{\partial x_{cs}^\ell}{\partial\alpha} \\ &\times\left(\frac{M_e}{M_c}\frac{x_{es}^k x_{es}^\ell}{r_{es}^5} + \frac{M_m}{M_c}\frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5}\right) \quad , \quad k = 1, 2, 3 \quad . \end{aligned} \tag{V-27}$$

The expressions for $\partial R^k/\partial\alpha$ and $\partial S^k/\partial\alpha$ are given in formulas (V-9) through (V-17), with p replaced by c. Finally, differentiating (III-9) with respect to α, we see that

$$\begin{aligned} \frac{\partial\Phi^k}{\partial\alpha} = \gamma M_s \sum_{j=1}^{N}\frac{M_j}{M_s}&\left[3\sum_{\ell=1}^{3}\frac{\partial x_{cs}^\ell}{\partial\alpha}\left(\frac{M_e}{M_c}\frac{x_{je}^k x_{je}^\ell}{r_{je}^5} + \frac{M_m}{M_c}\frac{x_{jm}^k x_{jm}^\ell}{r_{jm}^5}\right)\right. \\ &\left. - \frac{\partial x_{cs}^k}{\partial\alpha}\left(\frac{M_e}{M_c}\frac{1}{r_{je}^3} + \frac{M_m}{M_c}\frac{1}{r_{jm}^3}\right)\right] \quad , \quad k = 1, 2, 3 \end{aligned} \tag{V-28}$$

where we have made assumptions analogous to those of (V-6), namely,

$$\left.\begin{aligned} \frac{\partial x_{js}^k}{\partial\alpha} &= 0 \\ \frac{\partial x_{je}^k}{\partial\alpha} &= \frac{\partial x_{js}^k}{\partial\alpha} - \frac{\partial x_{es}^k}{\partial\alpha} = -\frac{\partial x_{cs}^k}{\partial\alpha} \\ \frac{\partial x_{jm}^k}{\partial\alpha} &= \frac{\partial x_{js}^k}{\partial\alpha} - \frac{\partial x_{ms}^k}{\partial\alpha} = -\frac{\partial x_{cs}^k}{\partial\alpha} \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-29}$$

If $\alpha = M_i/M_s$, we must add the following term to expression (V-28) for $\partial\Phi^k/\partial\alpha$:

$$\gamma M_s\left(\frac{M_e}{M_c}\frac{x_{ie}^k}{r_{ie}^3} + \frac{M_m}{M_c}\frac{x_{im}^k}{r_{im}^3} - \frac{x_{is}^k}{r_{is}^3}\right) \quad , \quad k = 1, 2, 3 \quad . \tag{V-30}$$

For the initial conditions $(\beta_m^1, \ldots, \beta_m^6)$ of (V-24), we assume that $\partial x_{cs}^k/\partial\beta_m^j = 0$. By (III-6), this implies that

$$\left.\begin{aligned} \frac{\partial x_{es}^k}{\partial\beta_m^j} &= -\frac{M_m}{M_c}\frac{\partial x_{me}^k}{\partial\beta_m^j} \\ \frac{\partial x_{ms}^k}{\partial\beta_m^j} &= \frac{M_e}{M_c}\frac{\partial x_{me}^k}{\partial\beta_m^j} \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-31}$$

We now determine the quantities on the right of (V-21) for $\alpha = \beta_m^j$. First, of course, $\partial(\gamma M_s)/\partial\beta_m^j = 0$ and $(\partial/\partial\beta_m^j)(M_c/M_s) = 0$. Next, differentiating (III-11) with respect to β_m^j and using (V-31), we see that

$$\begin{aligned} \frac{\partial B^k}{\partial\beta_m^j} = \gamma M_s \Bigg[3 \sum_{\ell=1}^{3} \frac{\partial x_{me}^\ell}{\partial\beta_m^j} &\left(\frac{M_m}{M_c}\frac{x_{es}^k x_{es}^\ell}{r_{es}^5} + \frac{M_e}{M_c}\frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5} \right) \\ &- \frac{\partial x_{me}^k}{\partial\beta_m^j}\left(\frac{M_m}{M_c}\frac{1}{r_{es}^3} + \frac{M_e}{M_c}\frac{1}{r_{ms}^3} \right) \Bigg] \quad , \quad k = 1, 2, 3 \end{aligned} \tag{V-32}$$

$$\begin{aligned} \frac{\partial \Psi^k}{\partial\beta_m^j} = \gamma M_s \sum_{i=1}^{N} \frac{M_i}{M_s} \Bigg[3 \sum_{\ell=1}^{3} \frac{\partial x_{me}^\ell}{\partial\beta_m^j} &\left(\frac{M_m}{M_c}\frac{x_{ie}^k x_{ie}^\ell}{r_{ie}^5} + \frac{M_e}{M_c}\frac{x_{im}^k x_{im}^\ell}{r_{im}^5} \right) \\ &- \frac{\partial x_{me}^k}{\partial\beta_m^j}\left(\frac{M_m}{M_c}\frac{1}{r_{ie}^3} + \frac{M_e}{M_c}\frac{1}{r_{im}^3} \right) \Bigg] \quad , \quad k = 1, 2, 3 \quad . \end{aligned} \tag{V-33}$$

In (V-33) we have assumed that

$$\left.\begin{aligned} \frac{\partial x_{is}^k}{\partial\beta_m^j} &= 0 \\ \frac{\partial x_{ie}^k}{\partial\beta_m^j} &= \frac{\partial x_{is}^k}{\partial\beta_m^j} - \frac{\partial x_{es}^k}{\partial\beta_m^j} = \frac{M_m}{M_c}\frac{\partial x_{me}^k}{\partial\beta_m^j} \\ \frac{\partial x_{im}^k}{\partial\beta_m^j} &= \frac{\partial x_{is}^k}{\partial\beta_m^j} - \frac{\partial x_{ms}^k}{\partial\beta_m^j} = -\frac{M_e}{M_c}\frac{\partial x_{me}^k}{\partial\beta_m^j} \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-34}$$

Finally, differentiating (III-79) with respect to β_m^j, we see that

$$\begin{aligned} \frac{\partial H^k}{\partial\beta_m^j} = \frac{\gamma M_s}{r_{me}^2}\frac{M_c}{M_s}\Bigg\{ \left(\frac{R_e}{r_{me}}\right)^2 \frac{J_2}{R_e^2} \Bigg[\frac{1}{r_{me}} &\left(\frac{\partial x_{me}^k}{\partial\beta_m^j} - \frac{5x_{me}^k}{r_{me}^2}\sum_{\ell=1}^{3} x_{me}^\ell \right. \\ &\left. \times \frac{\partial x_{me}^\ell}{\partial\beta_m^j} \right) \left(\frac{15}{2}\cos^2\Theta - \frac{3}{2}\right) + \frac{\partial(\cos\Theta)}{\partial\beta_m^j}\left(\frac{15x_{me}^k}{r_{me}}\cos\Theta - 3A_{3k} \right) \Bigg] \end{aligned}$$

$$+\left(\frac{R_m}{r_{me}}\right)^2 \frac{I_3}{M_m R_m^2} \sum_{i=2}^{3} \frac{I_i - I_1}{I_3} \left[\frac{1}{r_{me}}\left(\frac{\partial x_{me}^k}{\partial \beta_m^j} - \frac{5x_{me}^k}{r_{me}^2} \sum_{\ell=1}^{3} x_{me}^{\ell} \frac{\partial x_{me}^{\ell}}{\partial \beta_m^j}\right.\right.$$

$$\left.\left.\times\left(\frac{15}{2} D_i^2 - \frac{3}{2}\right) + \frac{\partial D_i}{\partial \beta_m^j}\left(\frac{15x_{me}^k}{r_{me}} D_i - 3B_{ik}\right)\right] + \left(\frac{R_e}{r_{me}}\right)^3 \frac{J_3}{R_e^3}$$

$$\times\left[\frac{1}{r_{me}}\left(\frac{\partial x_{me}^k}{\partial \beta_m^j} - \frac{6x_{me}^k}{r_{me}^2} \sum_{\ell=1}^{3} x_{me}^{\ell} \frac{\partial x_{me}^{\ell}}{\partial \beta_m^j}\right)\left(\frac{35}{2}\cos^3\Theta - \frac{15}{2}\cos\Theta\right)\right.$$

$$\left.\left.+ \frac{\partial(\cos\Theta)}{\partial \beta_m^j}\left(\frac{x_{me}^k}{r_{me}}\left(\frac{105}{2}\cos^2\Theta - \frac{15}{2}\right) - 15A_{3k}\cos\Theta\right)\right]\right] \quad , \quad k = 1, 2, 3 \qquad \text{(V-35)}$$

where, by (III-80),

$$\left.\begin{aligned} \frac{\partial(\cos\Theta)}{\partial \beta_m^j} &= \frac{1}{r_{me}} \sum_{\ell=1}^{3} A_{3\ell}\left(\frac{\partial x_{me}^{\ell}}{\partial \beta_m^j} - \frac{x_{me}^{\ell}}{r_{me}^2} \sum_{h=1}^{3} x_{me}^h \frac{\partial x_{me}^h}{\partial \beta_m^j}\right) \\ \frac{\partial D_i}{\partial \beta_m^j} &= \frac{1}{r_{me}} \sum_{\ell=1}^{3} B_{i\ell}\left(\frac{\partial x_{me}^{\ell}}{\partial \beta_m^j} - \frac{x_{me}^{\ell}}{r_{me}^2} \sum_{h=1}^{3} x_{me}^h \frac{\partial x_{me}^h}{\partial \beta_m^j}\right) \end{aligned}\right\} \quad i = 2, 3 \ . \qquad \text{(V-36)}$$

We now consider the parameters $M_{cs} = M_c/M_s$ and $M_{mc} = M_m/M_c$. Regarding these as independent parameters, formulas (III-6) imply that

$$\frac{\partial(M_c/M_s)}{\partial M_{cs}} = 1 \qquad \frac{\partial(M_c/M_s)}{\partial M_{mc}} = 0 \qquad \text{(V-37)}$$

$$\frac{\partial(M_m/M_c)}{\partial M_{cs}} = 0 \qquad \frac{\partial(M_m/M_c)}{\partial M_{mc}} = 1 \qquad \text{(V-38)}$$

$$\frac{\partial(M_e/M_c)}{\partial M_{cs}} = 0 \qquad \frac{\partial(M_e/M_c)}{\partial M_{mc}} = -1 \qquad \text{(V-39)}$$

$$\left.\begin{aligned} \frac{\partial x_{es}^k}{\partial M_{cs}} &= \frac{\partial x_{cs}^k}{\partial M_{cs}} - \frac{M_m}{M_c} \frac{\partial x_{me}^k}{\partial M_{cs}} \\ \frac{\partial x_{ms}^k}{\partial M_{cs}} &= \frac{\partial x_{cs}^k}{\partial M_{cs}} + \frac{M_e}{M_c} \frac{\partial x_{me}^k}{\partial M_{cs}} \end{aligned}\right\} \quad k = 1, 2, 3 \qquad \text{(V-40)}$$

$$\left.\begin{aligned} \frac{\partial x_{es}^k}{\partial M_{mc}} &= \frac{\partial x_{cs}^k}{\partial M_{mc}} - \frac{M_m}{M_c} \frac{\partial x_{me}^k}{\partial M_{mc}} - x_{me}^k \\ \frac{\partial x_{ms}^k}{\partial M_{mc}} &= \frac{\partial x_{cs}^k}{\partial M_{mc}} + \frac{M_e}{M_c} \frac{\partial x_{me}^k}{\partial M_{mc}} - x_{me}^k \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \qquad \text{(V-41)}$$

The quantities on the right-hand sides of (V-20) and (V-21) for $\alpha = M_{cs}$ are as follows. First, of course, $\partial(\gamma M_s)/\partial M_{cs} = 0$ and $(\partial/\partial M_{cs})(M_c/M_s) = 1$. Next, we have

$$\frac{\partial}{\partial M_{cs}}\left(\frac{M_e}{M_c}\frac{x_{es}^k}{r_{es}^3} + \frac{M_m}{M_c}\frac{x_{ms}^k}{r_{ms}^3}\right) = \frac{\partial x_{cs}^k}{\partial M_{cs}}\left(\frac{M_e}{M_c}\frac{1}{r_{es}^3} + \frac{M_m}{M_c}\frac{1}{r_{ms}^3}\right)$$
$$- 3\sum_{\ell=1}^{3}\frac{\partial x_{cs}^\ell}{\partial M_{cs}}\left(\frac{M_e}{M_c}\frac{x_{es}^k x_{es}^\ell}{r_{es}^5} + \frac{M_m}{M_c}\frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5}\right) + \frac{M_e}{M_c}\frac{M_m}{M_c}$$
$$\times\left[\frac{\partial x_{me}^k}{\partial M_{cs}}\left(\frac{1}{r_{ms}^3} - \frac{1}{r_{es}^3}\right) + 3\sum_{\ell=1}^{3}\frac{\partial x_{me}^\ell}{\partial M_{cs}}\right.$$
$$\left.\times\left(\frac{x_{es}^k x_{es}^\ell}{r_{es}^5} - \frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5}\right)\right] \quad , \quad k = 1, 2, 3 \quad . \tag{V-42}$$

The expressions for $\partial R^k/\partial M_{cs}$ and $\partial S^k/\partial M_{cs}$ are given in formulas (V-9) through (V-17), with p replaced by c. Differentiating (III-9) with respect to M_{cs}, we see that

$$\frac{\partial \Phi^k}{\partial M_{cs}} = \gamma M_s \sum_{j=1}^{N}\frac{M_j}{M_s}\left\{3\sum_{\ell=1}^{3}\frac{\partial x_{cs}^\ell}{\partial M_{cs}}\left(\frac{M_e}{M_c}\frac{x_{je}^k x_{je}^\ell}{r_{je}^5} + \frac{M_m}{M_c}\frac{x_{jm}^k x_{jm}^\ell}{r_{jm}^5}\right)\right.$$
$$- \frac{\partial x_{cs}^k}{\partial M_{cs}}\left(\frac{M_e}{M_c}\frac{1}{r_{je}^3} + \frac{M_m}{M_c}\frac{1}{r_{jm}^3}\right) + \frac{M_e}{M_c}\frac{M_m}{M_c}\left[\frac{\partial x_{me}^k}{\partial M_{cs}}\left(\frac{1}{r_{je}^3} - \frac{1}{r_{jm}^3}\right)\right.$$
$$\left.\left.- 3\sum_{\ell=1}^{3}\frac{\partial x_{me}^\ell}{\partial M_{cs}}\left(\frac{x_{je}^k x_{je}^\ell}{r_{je}^5} - \frac{x_{jm}^k x_{jm}^\ell}{r_{jm}^5}\right)\right]\right\} \quad , \quad k = 1, 2, 3 \tag{V-43}$$

where we have assumed that

$$\left.\begin{aligned}
&\frac{\partial x_{js}^k}{\partial M_{cs}} = 0 \\
&\frac{\partial x_{je}^k}{\partial M_{cs}} = \frac{\partial x_{js}^k}{\partial M_{cs}} - \frac{\partial x_{es}^k}{\partial M_{cs}} = -\frac{\partial x_{cs}^k}{\partial M_{cs}} + \frac{M_m}{M_c}\frac{\partial x_{me}^k}{\partial M_{cs}} \\
&\frac{\partial x_{jm}^k}{\partial M_{cs}} = \frac{\partial x_{js}^k}{\partial M_{cs}} - \frac{\partial x_{ms}^k}{\partial M_{cs}} = -\frac{\partial x_{cs}^k}{\partial M_{cs}} - \frac{M_e}{M_c}\frac{\partial x_{me}^k}{\partial M_{cs}}
\end{aligned}\right\} \quad k = 1, 2, 3 \quad . \tag{V-44}$$

Differentiating (III-11) with respect to M_{cs}, we see that

$$\frac{\partial B^k}{\partial M_{cs}} = \gamma M_s \left[\frac{\partial x^k_{cs}}{\partial M_{cs}}\left(\frac{1}{r^3_{es}} - \frac{1}{r^3_{ms}}\right) - 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{cs}}{\partial M_{cs}}\left(\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} - \frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\right)\right.$$

$$- \frac{\partial x^k_{me}}{\partial M_{cs}}\left(\frac{M_m}{M_c}\frac{1}{r^3_{es}} + \frac{M_e}{M_c}\frac{1}{r^3_{ms}}\right) + 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{me}}{\partial M_{cs}}\left(\frac{M_m}{M_c}\frac{x^k_{es}x^\ell_{es}}{r^5_{es}}\right.$$

$$\left.\left. + \frac{M_e}{M_c}\frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\right)\right] \quad , \quad k = 1, 2, 3 \qquad \text{(V-45)}$$

$$\frac{\partial \Psi^k}{\partial M_{cs}} = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s}\left[\frac{\partial x^k_{cs}}{\partial M_{cs}}\left(\frac{1}{r^3_{je}} - \frac{1}{r^3_{jm}}\right) - 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{cs}}{\partial M_{cs}}\right.$$

$$\times\left(\frac{x^k_{je}x^\ell_{je}}{r^5_{je}} - \frac{x^k_{jm}x^\ell_{jm}}{r^5_{jm}}\right) - \frac{\partial x^k_{me}}{\partial M_{cs}}\left(\frac{M_m}{M_c}\frac{1}{r^3_{je}} + \frac{M_e}{M_c}\frac{1}{r^3_{jm}}\right)$$

$$\left. + 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{me}}{\partial M_{cs}}\left(\frac{M_m}{M_c}\frac{x^k_{je}x^\ell_{je}}{r^5_{je}} + \frac{M_e}{M_c}\frac{x^k_{jm}x^\ell_{jm}}{r^5_{jm}}\right)\right] \quad , \quad k = 1, 2, 3 \qquad \text{(V-46)}$$

where we have used (V-44) in deriving (V-46). Let H^k_c denote the right side of (V-35), with partial derivatives with respect to β^j_m replaced by partial derivatives with respect to M_{cs}. Then, differentiating (III-79) with respect to M_{cs}, we see that

$$\frac{\partial H^k}{\partial M_{cs}} = H^k_c + \frac{H^k}{(M_c/M_s)} \quad , \quad k = 1, 2, 3 \quad . \qquad \text{(V-47)}$$

The quantities on the right-hand sides of (V-20) and (V-21) for $\alpha = M_{mc}$ are as follows. First, of course, $\partial(\gamma M_s)/\partial M_{mc} = 0$, and $(\partial/\partial M_{mc})\,(M_c/M_s) = 0$. Next, we have

$$\frac{\partial}{\partial M_{mc}}\left(\frac{M_e}{M_c}\frac{x^k_{es}}{r^3_{es}} + \frac{M_m}{M_c}\frac{x^k_{ms}}{r^3_{ms}}\right) = \left(\frac{x^k_{ms}}{r^3_{ms}} - \frac{x^k_{es}}{r^3_{es}}\right) + \left(\frac{\partial x^k_{cs}}{\partial M_{mc}} - x^k_{me}\right)$$

$$\times\left(\frac{M_e}{M_c}\frac{1}{r^3_{es}} + \frac{M_m}{M_c}\frac{1}{r^3_{ms}}\right) - 3 \sum_{\ell=1}^{3}\left(\frac{\partial x^\ell_{cs}}{\partial M_{mc}} - x^k_{me}\right)$$

$$\times\left(\frac{M_e}{M_c}\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} + \frac{M_m}{M_c}\frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\right) + \frac{M_e}{M_c}\frac{M_m}{M_c}\left[\frac{\partial x^k_{me}}{\partial M_{mc}}\right.$$

$$\left.\times\left(\frac{1}{r^3_{ms}} - \frac{1}{r^3_{es}}\right) + 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{me}}{\partial M_{mc}}\left(\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} - \frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\right)\right] \quad , \quad k = 1, 2, 3 \ . \ \text{(V-48)}$$

The expressions for $\partial R^k/\partial M_{mc}$ and $\partial S^k/\partial M_{mc}$ are given in formulas (V-9) through (V-17), with p replaced by c. Differentiating (III-9) with respect to M_{mc}, we see that

$$\frac{\partial \Phi^k}{\partial M_{mc}} = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \left\{ \left(\frac{x_{jm}^k}{r_{jm}^3} - \frac{x_{je}^k}{r_{je}^3} \right) + 3 \sum_{\ell=1}^{3} \left(\frac{\partial x_{cs}^\ell}{\partial M_{mc}} - x_{me}^\ell \right) \right.$$

$$\times \left(\frac{M_e}{M_c} \frac{x_{je}^k x_{je}^\ell}{r_{je}^5} + \frac{M_m}{M_c} \frac{x_{jm}^k x_{jm}^\ell}{r_{jm}^5} \right) - \left(\frac{\partial x_{cs}^k}{\partial M_{mc}} - x_{me}^k \right) \left(\frac{M_e}{M_c} \frac{1}{r_{je}^3} \right.$$

$$\left. + \frac{M_m}{M_c} \frac{1}{r_{jm}^3} \right) + \frac{M_e}{M_c} \frac{M_m}{M_c} \left[\frac{\partial x_{me}^k}{\partial M_{mc}} \left(\frac{1}{r_{je}^3} - \frac{1}{r_{jm}^3} \right) - 3 \sum_{\ell=1}^{3} \frac{\partial x_{me}^\ell}{\partial M_{mc}} \right.$$

$$\left. \left. \times \left(\frac{x_{je}^k x_{je}^\ell}{r_{je}^5} - \frac{x_{jm}^k x_{jm}^\ell}{r_{jm}^5} \right) \right] \right\} \quad , \quad k = 1, 2, 3 \qquad \text{(V-49)}$$

where we have assumed that

$$\left. \begin{aligned} \frac{\partial x_{js}^k}{\partial M_{mc}} &= 0 \\ \frac{\partial x_{je}^k}{\partial M_{mc}} &= \frac{\partial x_{js}^k}{\partial M_{mc}} - \frac{\partial x_{es}^k}{\partial M_{mc}} = -\frac{\partial x_{cs}^k}{\partial M_{mc}} + \frac{M_m}{M_c} \frac{\partial x_{me}^k}{\partial M_{mc}} + x_{me}^k \\ \frac{\partial x_{jm}^k}{\partial M_{mc}} &= \frac{\partial x_{js}^k}{\partial M_{mc}} - \frac{\partial x_{ms}^k}{\partial M_{mc}} = -\frac{\partial x_{cs}^k}{\partial M_{mc}} - \frac{M_e}{M_c} \frac{\partial x_{me}^k}{\partial M_{mc}} + x_{me}^k \end{aligned} \right\} \quad k = 1, 2, 3 \quad . \qquad \text{(V-50)}$$

Differentiating (III-11) with respect to M_{mc}, we see that

$$\frac{\partial B^k}{\partial M_{mc}} = \gamma M_s \left[\left(\frac{\partial x_{cs}^k}{\partial M_{mc}} - x_{me}^k \right) \left(\frac{1}{r_{es}^3} - \frac{1}{r_{ms}^3} \right) - 3 \sum_{\ell=1}^{3} \left(\frac{\partial x_{cs}^\ell}{\partial M_{mc}} - x_{me}^\ell \right) \right.$$

$$\times \left(\frac{x_{es}^k x_{es}^\ell}{r_{es}^5} - \frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5} \right) - \frac{\partial x_{me}^k}{\partial M_{mc}} \left(\frac{M_m}{M_c} \frac{1}{r_{es}^3} + \frac{M_e}{M_c} \frac{1}{r_{ms}^3} \right)$$

$$\left. + 3 \sum_{\ell=1}^{3} \frac{\partial x_{me}^\ell}{\partial M_{mc}} \left(\frac{M_m}{M_c} \frac{x_{es}^k x_{es}^\ell}{r_{es}^5} + \frac{M_e}{M_c} \frac{x_{ms}^k x_{ms}^\ell}{r_{ms}^5} \right) \right] \quad , \quad k = 1, 2, 3 \qquad \text{(V-51)}$$

$$\frac{\partial \Psi^k}{\partial M_{mc}} = \gamma M_s \sum_{j=1}^{N} \frac{M_j}{M_s} \left[\left(\frac{\partial x^k_{cs}}{\partial M_{mc}} - x^k_{me} \right) \left(\frac{1}{r^3_{je}} - \frac{1}{r^3_{jm}} \right) - 3 \sum_{\ell=1}^{3} \left(\frac{\partial x^\ell_{cs}}{\partial M_{mc}} - x^\ell_{me} \right) \left(\frac{x^k_{je} x^\ell_{je}}{r^5_{je}} - \frac{x^k_{jm} x^\ell_{jm}}{r^5_{jm}} \right) - \frac{\partial x^k_{me}}{\partial M_{mc}} \left(\frac{M_m}{M_c} \frac{1}{r^3_{je}} + \frac{M_e}{M_c} \frac{1}{r^3_{jm}} \right) + 3 \sum_{\ell=1}^{3} \frac{\partial x^\ell_{me}}{\partial M_{mc}} \left(\frac{M_m}{M_c} \frac{x^k_{je} x^\ell_{je}}{r^5_{je}} + \frac{M_e}{M_c} \frac{x^k_{jm} x^\ell_{jm}}{r^5_{jm}} \right) \right] , \quad k = 1, 2, 3 \qquad \text{(V-52)}$$

where we have used (V-50) in deriving (V-52). Differentiating (III-79) with respect to M_{mc}, we see that $\partial H^k/\partial M_{mc}$ is given by (V-35), with partial derivatives with respect to β^j_m being replaced by partial derivatives with respect to M_{mc}.

We now consider the time variation parameter λ in the gravitational constant as given in (V-3). Formulas (III-6) imply that

$$\left.\begin{aligned} \frac{\partial x^k_{es}}{\partial \lambda} &= \frac{\partial x^k_{cs}}{\partial \lambda} - \frac{M_m}{M_c} \frac{\partial x^k_{me}}{\partial \lambda} \\ \frac{\partial x^k_{ms}}{\partial \lambda} &= \frac{\partial x^k_{cs}}{\partial \lambda} + \frac{M_e}{M_c} \frac{\partial x^k_{me}}{\partial \lambda} \end{aligned}\right\} \quad k = 1, 2, 3 \quad . \qquad \text{(V-53)}$$

The quantities on the right sides of (V-20) and (V-21) for $\alpha = \lambda$ are as follows. First, of course, $\partial(\gamma M_s)/\partial\lambda$ is given by (V-4) and $(\partial/\partial\lambda)(M_c/M_s) = 0$. Next, we have that $(\partial/\partial\lambda)\,[(M_e/M_c)\,(x^k_{es}/r^3_{es}) + (M_m/M_c)\,(x^k_{ms}/r^3_{ms})]$ is given by the right side of (V-42) with partial derivatives with respect to M_{cs} being replaced by partial derivatives with respect to λ. The expressions for $\partial R^k/\partial\lambda$ and $\partial S^k/\partial\lambda$ are given in formulas (V-9) through (V-17), with p replaced by c. If we let Φ^k_λ denote the right-hand side of (V-43), with the partial derivatives with respect to M_{cs} being replaced by partial derivatives with respect to λ, differentiation of (III-9) gives

$$\frac{\partial \Phi^k}{\partial \lambda} = \Phi^k_\lambda + (\gamma M_s)_o\,(t - t_o) \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{M_e}{M_c} \frac{x^k_{je}}{r^3_{je}} + \frac{M_m}{M_c} \frac{x^k_{jm}}{r^3_{jm}} - \frac{x^k_{js}}{r^3_{js}} \right) , \quad k = 1, 2, 3 \quad . \qquad \text{(V-54)}$$

If we let B^k_λ and Ψ^k_λ denote the right-hand sides of (V-45) and (V-46), respectively, with the partial derivatives with respect to M_{cs} being replaced by partial derivatives with respect to λ, differentiation of (III-11) gives

$$\frac{\partial B^k}{\partial \lambda} = B^k_\lambda + (\gamma M_s)_o\,(t - t_o) \left(\frac{x^k_{es}}{r^3_{es}} - \frac{x^k_{ms}}{r^3_{ms}} \right) , \quad k = 1, 2, 3 \qquad \text{(V-55)}$$

$$\frac{\partial \Psi^k}{\partial \lambda} = \Psi^k_\lambda + (\gamma M_s)_o\,(t - t_o) \sum_{j=1}^{N} \frac{M_j}{M_s} \left(\frac{x^k_{jm}}{r^3_{jm}} - \frac{x^k_{je}}{r^3_{je}} \right) , \quad k = 1, 2, 3 \quad . \qquad \text{(V-56)}$$

If we let H^k_λ denote the right-hand side of (V-35), with partial derivatives with respect to β^j_m being replaced by partial derivatives with respect to λ, differentiation of (III-79) gives

$$\frac{\partial H^k}{\partial \lambda} = H_\lambda^k + \frac{(\gamma M_s)_o (t - t_o)}{\gamma M_s} H^k \quad , \quad k = 1, 2, 3 \quad . \tag{V-57}$$

We have not derived the differential equations satisfied by the partial derivatives of the position and velocity of the Moon with respect to the higher harmonics of the gravitational potentials of the Earth and Moon, because the gravitational potential of the Earth has been determined quite accurately from the motion of artificial Earth satellites, and the gravitational potential of the Moon will be determined quite accurately in the near future from the motion of artificial lunar satellites.

VI. ENCKE'S EQUATIONS OF MOTION AND ENCKE'S EQUATIONS FOR PARTIAL DERIVATIVES WITH RESPECT TO INITIAL CONDITIONS

A. PLANET CASE

Let $(x_{ps}^1, \ldots, x_{ps}^6)$ denote the components of position and velocity of a planet relative to the Sun, satisfying (V-1) with initial conditions $(x_{ops}^1, \ldots, x_{ops}^6)$ and with the gravitational constant γM_s being given by (V-3). Let $(y_{ps}^1, \ldots, y_{ps}^6)$ be the solutions of the equations in (II-63) with $\mu = (\gamma M_s)_o \, [1 + (M_p/M_s)]$, and with initial conditions $(x_{ops}^1, \ldots, x_{ops}^6)$. The quantities $(y_{ps}^1, \ldots, y_{ps}^6)$ are the components of position and velocity in the elliptic orbit osculating to the true orbit of the planet at the initial time. Let

$$\xi_{ps}^k = x_{ps}^k - y_{ps}^k \quad , \quad k = 1, \ldots, 6 \quad . \tag{VI-1}$$

Subtracting (II-63) from (V-1), we see that the quantities $(\xi_{ps}^1, \ldots, \xi_{ps}^6)$ satisfy the system of equations

$$\left.\begin{aligned} \frac{d\xi_{ps}^k}{dt} &= \xi_{ps}^{k+3} \\ \frac{d\xi_{ps}^{k+3}}{dt} &= (\gamma M_s)_o \left(1 + \frac{M_p}{M_s}\right)\left[\left(\frac{1}{\rho_{ps}^3} - \frac{1}{r_{ps}^3}\right) y_{ps}^k - \frac{\xi_{ps}^k}{r_{ps}^3}\right. \\ &\qquad \left. - \lambda(t - t_o)\, \frac{x_{ps}^k}{r_{ps}^3}\right] + \Omega^k + R^k + S^k + \frac{1}{M_p}\, F_p^k \\ \xi_{ps}^k &= \xi_{ps}^{k+3} = 0 \quad \text{when } t = t_o \end{aligned}\right\} \quad k = 1, 2, 3 \tag{VI-2}$$

where $\rho_{ps} = \sqrt{(y_{ps}^1)^2 + (y_{ps}^2)^2 + (y_{ps}^3)^2}$. The quantities $(y_{ps}^1, \ldots, y_{ps}^6)$ are known as functions of time from the formulas in Sec. II-B, so that if we numerically integrate (VI-2) to find $(\xi_{ps}^1, \ldots, \xi_{ps}^6)$, the position and velocity of the planet $(x_{ps}^1, \ldots, x_{ps}^6)$ can be determined from (VI-1).

Let $\partial x_{ps}^k/\partial \beta_p^j$ (j, k = 1, . . . , 6) denote the partial derivatives of the position and velocity of the planet relative to the Sun with respect to the initial osculating orbital elements $(\beta_p^1, \ldots, \beta_p^6) = (a, e, i, \Omega, \omega, \ell_o)$. These quantities satisfy differential equations (V-2) with initial conditions $\partial x_{ops}^k/\partial \beta_p^j$ (j, k = 1, . . . , 6). Let $\partial y_{ps}^k/\partial \beta_p^j$ (j, k = 1, . . . , 6) be the solutions of differential equations (II-64) with these same initial conditions and with $\mu = (\gamma M_s)_o \, [1 + (M_p/M_s)]$. We define

$$\eta_{pj}^k = \frac{\partial x_{ps}^k}{\partial \beta_p^j} - \frac{\partial y_{ps}^k}{\partial \beta_p^j} \quad , \quad j, k = 1, \ldots, 6 \quad . \tag{VI-3}$$

Subtracting (II-64) from (V-2), we see that

$$
\left.
\begin{aligned}
&\frac{d\eta_{pj}^{k}}{dt} = \eta_{pj}^{k+3} \\
&\frac{d\eta_{pj}^{k+3}}{dt} = (\gamma M_s)_o \left(1 + \frac{M_p}{M_s}\right) \left[\left(\frac{1}{\rho_{ps}^3} - \frac{1}{r_{ps}^3}\right) \frac{\partial y_{ps}^k}{\partial \beta_p^j} - \frac{\eta_{pj}^k}{r_{ps}^3}\right. \\
&\qquad + 3 \sum_{\ell=1}^{3} \frac{\partial y_{ps}^{\ell}}{\partial \beta_p^j} \left(\frac{x_{ps}^k x_{ps}^{\ell}}{r_{ps}^5} - \frac{y_{ps}^k y_{ps}^{\ell}}{\rho_{ps}^5}\right) + \frac{3x_{ps}^k}{r_{ps}^5} \sum_{\ell=1}^{3} x_{ps}^{\ell} \eta_{pj}^{\ell} \\
&\qquad \left. + \frac{\lambda(t - t_o)}{r_{ps}^3} \left(\frac{3x_{ps}^k}{r_{ps}^2} \sum_{\ell=1}^{3} x_{ps}^{\ell} \frac{\partial x_{ps}^{\ell}}{\partial \beta_p^j} - \frac{\partial x_{ps}^k}{\partial \beta_p^j}\right)\right] \\
&\qquad + \frac{\partial \Omega^k}{\partial \beta_p^j} + \frac{\partial R^k}{\partial \beta_p^j} + \frac{\partial S^k}{\partial \beta_p^j} + \frac{\partial}{\partial \beta_p^j} \left(\frac{1}{M_p} F_p^k\right) \\
&\eta_{pj}^{k} = \eta_{pj}^{k+3} = 0 \quad \text{when } t = t_o
\end{aligned}
\right\}
\quad
\begin{aligned}
k &= 1, 2, 3 \\
j &= 1, \ldots, 6
\end{aligned}
\quad . \qquad \text{(VI-4)}
$$

The quantities $\partial y_{ps}^k/\partial \beta_p^j$ are known as functions of time from the formulas in Sec. II-D, so that if we numerically integrate (VI-4) to find the η_{pj}^k, the partial derivatives of the position and velocity of the planet with respect to the initial osculating orbital elements $\partial x_{ps}^k/\partial \beta_p^j$ can be determined from (VI-3).

If the quantities ξ_{ps}^k get too large as time progresses, a new osculating elliptic orbit can be chosen and the integration of system (VI-2) can commence again with initial conditions zero. If we are also integrating system (VI-4) and desire to change Encke orbits, the following procedure must be followed. Let $(\beta^1, \ldots, \beta^6)$ be the osculating elliptic orbital elements at the initial time t_o; let $(x_*^1, \ldots, x_*^6)$ and $\partial x_*^k/\partial \beta^j$ (j, k = 1, . . . , 6) be the position, velocity and partial derivatives with respect to initial conditions at the time t_* at which we wish to change Encke orbits. These quantities are known from the numerical integration of the equations of motion and the equations for the partial derivatives with respect to initial conditions from time t_o to time t_*. Let $(\beta_*^1, \ldots, \beta_*^6)$ be the osculating elliptic orbital elements at time t_* determined from the formulas in Sec. II-C. Integration of (VI-2) and (VI-4) with initial conditions zero from time t_* to time t determines the position and velocity of the planet $(x^1, \ldots, x^6)$ and the partial derivatives of position and velocity with respect to the orbital elements at time t_*, $\partial x^k/\partial \beta_*^j$ (j, k = 1, . . . , 6). Then, to determine the partial derivatives with respect to the orbital elements at time t_o, we must use the relation

$$
\frac{\partial x^k}{\partial \beta^j} = \sum_{\ell=1}^{6} \left(\sum_{i=1}^{6} \frac{\partial x_*^i}{\partial \beta^j} \frac{\partial \beta_*^{\ell}}{\partial x_*^i}\right) \frac{\partial x^k}{\partial \beta_*^{\ell}} \quad , \quad j, k = 1, \ldots, 6 \qquad \text{(VI-5)}
$$

where the matrix $\partial \beta_*^{\ell}/\partial x_*^i$ is determined from the formulas in Sec. II-E.

The elliptic orbit position and velocity in the new Encke orbit osculating to the true orbit at time t_* satisfy differential equations (II-63) with $\mu = (\gamma M_s)_* (1 + M_p/M_s)$, where

$$
(\gamma M_s)_* = (\gamma M_s)_o [1 + \lambda(t_* - t_o)] \quad . \qquad \text{(VI-6)}
$$

Thus, the factor $(\gamma M_s)_o$ in (VI-2) with initial conditions zero at time t_* must be replaced by $(\gamma M_s)_*$, and the term $\lambda(t - t_o)$ replaced by a term $G(t - t_*)$ satisfying

$$\gamma M_s = (\gamma M_s)_* \, [1 + G(t - t_*)] \tag{VI-7}$$

which implies that

$$G = \frac{\lambda}{1 + \lambda(t_* - t_o)} \quad . \tag{VI-8}$$

Exactly similar comments apply to (VI-4).

B. EARTH-MOON CASE

Let $(x^1_{cs}, \ldots, x^6_{cs})$ and $(x^1_{me}, \ldots, x^6_{me})$ denote the components of position and velocity of the Earth-Moon barycenter relative to the Sun and of the Moon relative to the Earth, satisfying (V-18) and (V-19) with initial conditions $(x^1_{ocs}, \ldots, x^6_{ocs})$ and $(x^1_{ome}, \ldots, x^6_{ome})$. We assume that the gravitational constant γM_s is given by (V-3). Let $(y^1_{cs}, \ldots, y^6_{cs})$ and $(y^1_{me}, \ldots, y^6_{me})$ denote the components of position and velocity in the elliptic orbits osculating at the initial time to the true orbits of the Earth-Moon barycenter and of the Moon. The quantities $(y^1_{cs}, \ldots, y^6_{cs})$ are the solutions of the equations in (II-63) with $\mu = (\gamma M_s)_o \, [1 + (M_c/M_s)]$ and with initial conditions $(x^1_{ocs}, \ldots, x^6_{ocs})$, while the quantities $(y^1_{me}, \ldots, y^6_{me})$ are the solutions of (II-63) with $\mu = (\gamma M_s)_o \, (M_c/M_s)$ and with initial conditions $(x^1_{ome}, \ldots, x^6_{ome})$. Let

$$\left.\begin{aligned} \xi^k_{cs} &= x^k_{cs} - y^k_{cs} \\ \xi^k_{me} &= x^k_{me} - y^k_{me} \end{aligned}\right\} \quad k = 1, \ldots, 6 \quad . \tag{VI-9}$$

Subtracting (II-63) from (V-18) and (V-19), we see that the quantities $(\xi^1_{cs}, \ldots, \xi^6_{cs})$ and $(\xi^1_{me}, \ldots, \xi^6_{me})$ satisfy the system of equations

$$\left.\begin{aligned} \frac{d\xi^k_{cs}}{dt} &= \xi^{k+3}_{cs} \\ \frac{d\xi^{k+3}_{cs}}{dt} &= (\gamma M_s)_o \left(1 + \frac{M_c}{M_s}\right) \Bigg\{ \left[\frac{1}{\rho^3_{cs}} - \left(\frac{M_e}{M_c}\frac{1}{r^3_{es}} + \frac{M_m}{M_c}\frac{1}{r^3_{ms}}\right)\right] y^k_{cs} \\ &\quad - \left(\frac{M_e}{M_c}\frac{1}{r^3_{es}} + \frac{M_m}{M_c}\frac{1}{r^3_{ms}}\right)\xi^k_{cs} + \frac{M_e}{M_c}\frac{M_m}{M_c}\left(\frac{1}{r^3_{es}} - \frac{1}{r^3_{ms}}\right) x^k_{me} \\ &\quad - \lambda(t - t_o)\left(\frac{M_e}{M_c}\frac{x^k_{es}}{r^3_{es}} + \frac{M_m}{M_c}\frac{x^k_{ms}}{r^3_{ms}}\right)\Bigg\} + \Phi^k + R^k + S^k \\ &\quad + \frac{1}{M_c}\left(F^k_e + F^k_m\right) \\ \xi^k_{cs} &= \xi^{k+3}_{cs} = 0 \quad \text{when } t = t_o \end{aligned}\right\} \quad k = 1, 2, 3 \tag{VI-10}$$

$$\left.\begin{aligned}
&\frac{d\xi^k_{me}}{dt} = \xi^{k+3}_{me}\\
&\frac{d\xi^{k+3}_{me}}{dt} = (\gamma M_s)_o \frac{M_c}{M_s}\left[\left(\frac{1}{\rho^3_{me}} - \frac{1}{r^3_{me}}\right) y^k_{me} - \frac{\xi^k_{me}}{r^3_{me}} - \lambda(t - {}_o)\frac{x^k_{me}}{r^3_{me}}\right]\\
&\qquad + B^k + \Psi^k + H^k + \left(\frac{1}{M_m} F^k_m - \frac{1}{M_e} F^k_e\right)\\
&\xi^k_{me} = \xi^{k+3}_{me} = 0 \quad \text{when } t = t_o
\end{aligned}\right\} \quad k = 1, 2, 3 \qquad \text{(VI-11)}$$

where $\rho_{cs} = \sqrt{(y^1_{cs})^2 + (y^2_{cs})^2 + (y^3_{cs})^2}$ and $\rho_{me} = \sqrt{(y^1_{me})^2 + (y^2_{me})^2 + (y^3_{me})^2}$. The quantities $(y^1_{cs}, \ldots, y^6_{cs})$ and $(y^1_{me}, \ldots, y^6_{me})$ are known as functions of time from the formulas in Sec. II-B so that, if we numerically integrate (VI-10) and (VI-11) to find $(\xi^1_{cs}, \ldots, \xi^6_{cs})$ and $(\xi^1_{me}, \ldots, \xi^6_{me})$, the position and velocity of the Earth-Moon barycenter and of the Moon $(x^1_{cs}, \ldots, x^6_{cs})$ and $(x^1_{me}, \ldots, x^6_{me})$ can be determined from (VI-9).

Let $\partial x^k_{cs}/\partial\beta^j_c$ and $\partial x^k_{me}/\partial\beta^j_m$ (j, k = 1, ..., 6) denote the partial derivatives of the position and velocity of the Earth-Moon barycenter and of the Moon with respect to the initial osculating orbital elements $(\beta^1_c, \ldots, \beta^6_c)$ and $(\beta^1_m, \ldots, \beta^6_m)$. These quantities satisfy differential equations (V-20) and (V-21) with initial conditions $\partial x^k_{ocs}/\partial\beta^j_c$ and $\partial x^k_{ome}/\partial\beta^j_m$ (j, k = 1, ..., 6). Let $\partial y^k_{cs}/\partial\beta^j_c$ and $\partial y^k_{me}/\partial\beta^j_m$ (j, k = 1, ..., 6) be the solutions of differential equations (II-64) with these same initial conditions and with $\mu = (\gamma M_s)_o\,[1 + (M_c/M_s)]$ and $\mu = (\gamma M_s)_o\,(M_c/M_s)$, respectively. We define

$$\left.\begin{aligned}
\eta^k_{cj} &= \frac{\partial x^k_{cs}}{\partial\beta^j_c} - \frac{\partial y^k_{cs}}{\partial\beta^j_c}\\
\eta^k_{mj} &= \frac{\partial x^k_{me}}{\partial\beta^j_m} - \frac{\partial y^k_{me}}{\partial\beta^j_m}
\end{aligned}\right\} \quad j, k = 1, \ldots, 6 \quad . \qquad \text{(VI-12)}$$

Subtracting (II-64) from (V-20) and (V-21), we see that

$$
\left.
\begin{aligned}
\frac{d\eta^k_{cj}}{dt} &= \eta^{k+3}_{cj}\\
\frac{d\eta^{k+3}_{cj}}{dt} &= (\gamma M_s)_o \left(1+\frac{M_c}{M_s}\right)\biggl\{\biggl[\biggl[\frac{1}{\rho^3_{cs}} - \left(\frac{M_e}{M_c}\frac{1}{r^3_{es}} + \frac{M_m}{M_c}\frac{1}{r^3_{ms}}\right)\biggr]\\
&\quad\times \frac{\partial y^k_{cs}}{\partial\beta^j_c} - \left(\frac{M_e}{M_c}\frac{1}{r^3_{es}} + \frac{M_m}{M_c}\frac{1}{r^3_{ms}}\right)\eta^k_{cj}\\
&\quad -3\sum_{\ell=1}^{3} \frac{\partial y^\ell_{cs}}{\partial\beta^j_c}\biggl[\frac{y^k_{cs}y^\ell_{cs}}{\rho^5_{cs}} - \biggl(\frac{M_e}{M_c}\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} + \frac{M_m}{M_c}\\
&\quad\times\frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\biggr)\biggr] + 3\sum_{\ell=1}^{3} \eta^\ell_{cs}\biggl(\frac{M_e}{M_c}\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} + \frac{M_m}{M_c}\\
&\quad\times \frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\biggr) + \lambda(t-t_o)\biggl[3\sum_{\ell=1}^{3} \frac{\partial x^\ell_{cs}}{\partial\beta^j_c}\biggl(\frac{M_e}{M_c}\\
&\quad\times\frac{x^k_{es}x^\ell_{es}}{r^5_{es}} + \frac{M_m}{M_c}\frac{x^k_{ms}x^\ell_{ms}}{r^5_{ms}}\biggr) - \frac{\partial x^k_{cs}}{\partial\beta^j_c}\biggl(\frac{M_e}{M_c}\frac{1}{r^3_{es}}\\
&\quad+\frac{M_m}{M_c}\frac{1}{r^3_{ms}}\biggr)\biggr]\biggr]\biggr\} + \frac{\partial\Phi^k}{\partial\beta^j_c} + \frac{\partial R^k}{\partial\beta^j_c} + \frac{\partial S^k}{\partial\beta^j_c}\\
&\quad+\frac{\partial}{\partial\beta^j_c}\left[\frac{1}{M_c}\left(F^k_e+F^k_m\right)\right]\\
\eta^k_{cj} &= \eta^{k+3}_{cj} = 0 \quad \text{when } t = t_o
\end{aligned}
\right\}
\quad
\begin{aligned}
k &= 1,2,3\\
j &= 1,\ldots,6
\end{aligned}
\qquad \text{(VI-13)}
$$

$$
\left.
\begin{aligned}
&\frac{d\eta_{mj}^{k}}{dt} = \frac{d\eta_{mj}^{k+3}}{dt} \\
&\frac{d\eta_{mj}^{k+3}}{dt} = (\gamma M_s)_o \frac{M_c}{M_s}\left[\left(\frac{1}{\rho_{me}^{3}} - \frac{1}{r_{me}^{3}}\right)\frac{\partial y_{me}^{k}}{\partial \beta_m^{j}} - \frac{\eta_{mj}^{k}}{r_{me}^{3}}\right. \\
&\qquad + 3\sum_{\ell=1}^{3} \frac{\partial y_{me}^{\ell}}{\partial \beta_m^{j}}\left(\frac{x_{me}^{k}x_{me}^{\ell}}{r_{me}^{5}} - \frac{y_{me}^{k}y_{me}^{\ell}}{\rho_{me}^{5}}\right) \\
&\qquad + \frac{3x_{me}^{k}}{r_{me}^{5}}\sum_{\ell=1}^{3} x_{me}^{\ell}\eta_{me}^{\ell} + \frac{\lambda(t-t_o)}{r_{me}^{3}}\left(\frac{3x_{me}^{k}}{r_{me}^{2}}\right. \\
&\qquad \left.\left.\times \sum_{\ell=1}^{3} x_{me}^{\ell}\frac{\partial x_{me}^{\ell}}{\partial \beta_m^{j}} - \frac{\partial x_{me}^{k}}{\partial \beta_m^{j}}\right)\right] + \frac{\partial B^{k}}{\partial \beta_m^{j}} + \frac{\partial \Psi^{k}}{\partial \beta_m^{j}} \\
&\qquad + \frac{\partial H^{k}}{\partial \beta_m^{j}} + \frac{\partial}{\partial \beta_m^{j}}\left(\frac{1}{M_m}F_m^{k} - \frac{1}{M_e}F_e^{k}\right) \\
&\eta_{mj}^{k} = \eta_{mj}^{k+3} = 0 \quad \text{when } t = t_o
\end{aligned}
\right\}
\begin{aligned}
&k = 1, 2, 3 \\
&j = 1, \ldots, 6
\end{aligned}
\quad . \qquad \text{(VI-14)}
$$

The quantities $\partial y_{cs}^{k}/\partial \beta_c^{j}$ and $\partial y_{me}^{k}/\partial \beta_m^{j}$ are known as functions of time from the formulas in Sec. II-D so that, if we numerically integrate (VI-13) and (VI-14) to find the η_{ej}^{k} and η_{mj}^{k}, we can find the partial derivatives of position and velocity with respect to initial osculating orbital elements for the Earth-Moon barycenter and for the Moon from formulas (VI-12).

The method of changing Encke orbits for the Earth-Moon barycenter and for the Moon integrations is the same as discussed at the end of Sec. VI-A. It will be necessary to change Encke orbits more often in the case of the Moon than in the case of the Earth-Moon barycenter.

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APPENDIX A
PRECESSION-NUTATION OF THE EARTH

The notation 1950.0 denotes the beginning of the Besselian year, which is the instant near the beginning of the calendar year 1950 when the right ascension of the mean Sun was exactly 18^h40^m. In more conventional notation, 1950.0 is thus J.E.D. 2433282.423 or 1950 January $0^d.923$ Ephemeris Time (see Ref. 5).

Let (X^1, X^2, X^3) be a rectangular coordinate system whose X^1-axis points toward the mean vernal equinox of 1950.0, whose X^3-axis points toward the mean north pole of 1950.0, and whose X^2-axis completes the right-hand system. In more concise language, we say that (X^1, X^2, X^3) is a rectangular coordinate system referred to the mean equinox and equator of 1950.0 This is the coordinate system in which we are going to integrate the equations of motion. Let (x^1, x^2, x^3) be a coordinate system referred to the true equinox and equator of date with the same origin as the (X^1, X^2, X^3) coordinate system. The x^1-axis points toward the true vernal equinox of date, the x^3-axis points toward the true north pole of date, and the x^2-axis completes the right-hand system. The relation between the (x^1, x^2, x^3) and (X^1, X^2, X^3) coordinate systems is

$$\left.\begin{aligned} x^j &= \sum_{\ell=1}^{3} A_{j\ell} X^\ell \\ X^j &= \sum_{\ell=1}^{3} A_{\ell j} x^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{A-1}$$

where the orthogonal matrix A is given by

$$A_{ij} = \sum_{\ell=1}^{3} N_{i\ell} P_{\ell j} \quad . \quad i, j = 1, 2, 3 \tag{A-2}$$

with N and P being the nutation and precession matrices, respectively. The matrix A appears in formulas (III-55) and (III-79).

We now give the established expressions for the precession and nutation. First, we follow Ref. 37 in defining the angles

$$\begin{aligned} \zeta_o &= 2304''.948T + 0''.302T^2 + 0''.0179T^3 \\ z &= 2304''.948T + 1''.093T^2 + 0''.0192T^3 \\ \Theta &= 2004''.255T - 0''.426T^2 - 0''.0416T^3 \end{aligned} \tag{A-3}$$

where T is measured in tropical centuries of 36524.21988 ephemeris days from the epoch 1950.0 (J.E.D. 2433282.423) to the instant of interest. Then the precession matrix at this instant is given by[38]

$$P_{11} = \cos\zeta_o \cos\Theta \cos z - \sin\zeta_o \sin z$$

$$P_{12} = -\sin\zeta_o \cos\Theta \cos z - \cos\zeta_o \sin z$$

$$P_{13} = -\sin\Theta \cos z$$

$$P_{21} = \cos\zeta_o \cos\Theta \sin z + \sin\zeta_o \cos z$$

$$P_{22} = -\sin\zeta_o \cos\Theta \sin z + \cos\zeta_o \cos z$$

$$P_{23} = -\sin\Theta \sin z$$

$$P_{31} = \cos\zeta_o \sin\Theta$$

$$P_{32} = -\sin\zeta_o \sin\Theta$$

$$P_{33} = \cos\Theta \qquad \text{(A-4)}$$

The mean obliquity of the ecliptic is[39]

$$\epsilon_o = 23°27'08''.26 - 46''.845T - 0''.0059T^2 + 0''.00181T^3 \qquad \text{(A-5)}$$

where T is measured in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 E.T. = J.E.D. 2415020.0 to the instant of interest. Let $\Delta\psi$ and $\Delta\epsilon$ be the nutations in longitude and obliquity, respectively, as given by the series in Ref. 40. The true obliquity of the ecliptic is then

$$\epsilon = \epsilon_o + \Delta\epsilon \quad . \qquad \text{(A-6)}$$

Finally, the nutation matrix is given by[41]

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} = \begin{bmatrix} 1 & -\Delta\psi\cos\epsilon & -\Delta\psi\sin\epsilon \\ \Delta\psi\cos\epsilon & 1 & -\Delta\epsilon \\ \Delta\psi\sin\epsilon & \Delta\epsilon & 1 \end{bmatrix} \quad . \qquad \text{(A-7)}$$

APPENDIX B
ROTATION AND PHYSICAL LIBRATION OF THE MOON

Let (y^1, y^2, y^3) be the coordinate system with origin at the center of mass of the Moon whose coordinate axes point along the principal axes of inertia of the Moon. The y^3-axis points along the axis of rotation of the Moon, while the y^1-axis always points in the general direction of the Earth, the period of rotation of the Moon about its center of mass being the same as its orbital period. The y^2-axis completes the right-hand system, so that the (y^1, y^2) plane is the plane of the Moon's equator. In this coordinate system, the second harmonic of the gravitational potential of the Moon has the form (III-70).

Let (u^1, u^2, u^3) be the coordinate system with origin at the center of mass of the Moon referred to the mean equinox and ecliptic of date, and suppose that ψ is the longitude of the descending node of the lunar equator on the ecliptic of date measured from the mean equinox of date, that Θ is the inclination of the lunar equator on the ecliptic of date, and that φ is the angular distance of the positive part of the y^1-axis of the coordinate system (y^1, y^2, y^3) from the descending node of the lunar equator. Then (II-1) and (II-2), with $\Omega = \psi$, $i = -\Theta$ and $\omega = \varphi$, imply that

$$\left.\begin{aligned} y^j &= \sum_{\ell=1}^{3} U_{j\ell} u^\ell \\ u^j &= \sum_{\ell=1}^{3} U_{\ell j} y^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{B-1}$$

where

$$\begin{aligned}
U_{11} &= \cos\psi \cos\varphi - \sin\psi \sin\varphi \cos\Theta \\
U_{12} &= \sin\psi \cos\varphi + \cos\psi \sin\varphi \cos\Theta \\
U_{13} &= -\sin\varphi \sin\Theta \\
U_{21} &= -\cos\psi \sin\varphi - \sin\psi \cos\varphi \cos\Theta \\
U_{22} &= -\sin\psi \sin\varphi + \cos\psi \cos\varphi \cos\Theta \\
U_{23} &= -\cos\varphi \sin\Theta \\
U_{31} &= -\sin\psi \sin\Theta \\
U_{32} &= \cos\psi \sin\Theta \\
U_{33} &= \cos\Theta
\end{aligned} \tag{B-2}$$

Let (x^1, x^2, x^3) be the coordinate system with origin at the center of mass of the Moon referred to the mean equinox and equator (of the Earth) of date. Let ϵ_0 be the mean inclination of the ecliptic as given in (A-5). Then we can write[42]

$$\left.\begin{aligned} u^j &= \sum_{\ell=1}^{3} V_{j\ell} x^\ell \\ x^j &= \sum_{\ell=1}^{3} V_{\ell j} u^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{B-3}$$

where

$$\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon_o & \sin\epsilon_o \\ 0 & -\sin\epsilon_o & \cos\epsilon_o \end{bmatrix} \quad . \tag{B-4}$$

Combining (B-1) and (B-3), we see that

$$\left.\begin{aligned} y^j &= \sum_{\ell=1}^{3} W_{j\ell} x^\ell \\ x^j &= \sum_{\ell=1}^{3} W_{\ell j} y^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{B-5}$$

where

$$W_{j\ell} = \sum_{k=1}^{3} U_{jk} V_{k\ell} \quad , \quad j, \ell = 1, 2, 3 \tag{B-6}$$

so that, by (B-4), we have

$$\left.\begin{aligned} W_{j1} &= U_{j1} \\ W_{j2} &= U_{j2} \cos\epsilon_o - U_{j3} \sin\epsilon_o \\ W_{j3} &= U_{j2} \sin\epsilon_o + U_{j3} \cos\epsilon_o \end{aligned}\right\} \quad j = 1, 2, 3 \quad . \tag{B-7}$$

Let (X^1, X^2, X^3) be the coordinate system with origin at the center of mass of the Moon referred to the mean equinox and equator (of the Earth) of 1950.0, the reference system in which we are going to integrate the equations of motion. Then we have

$$\left.\begin{aligned} x^j &= \sum_{\ell=1}^{3} P_{j\ell} X^\ell \\ X^j &= \sum_{\ell=1}^{3} P_{\ell j} x^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{B-8}$$

where $(P_{j\ell})$ is the precession matrix of (A-4). Combining (B-6) and (B-8), we have

$$\left.\begin{aligned} y^j &= \sum_{\ell=1}^{3} B_{j\ell} X^\ell \\ X^j &= \sum_{\ell=1}^{3} B_{\ell j} y^\ell \end{aligned}\right\} \quad j = 1, 2, 3 \tag{B-9}$$

where

$$B_{j\ell} = \sum_{k=1}^{3} W_{jk} P_{k\ell} \quad , \quad j, \ell = 1, 2, 3 \quad . \tag{B-10}$$

This is the matrix B which appears in (III-75) and (III-79).

Let ☾ be the mean longitude of the Moon, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit and then along the orbit. Let Ω be the longitude of the mean ascending node of the lunar orbit on the ecliptic measured from the mean equinox of date. Finally, let I be the inclination of the mean lunar equator to the ecliptic. Then the angles ψ, Θ, φ of formulas (B-2) are[43]

$$\begin{aligned} \psi &= \Omega + \sigma \\ \Theta &= I + \rho \\ \varphi &= 180° + (☾ - \Omega) + (\tau - \sigma) \end{aligned} \tag{B-11}$$

where σ, ρ and τ are the physical librations in node, inclination and longitude, respectively.

We now determine the quantities on the right side of (B-11). First, the inclination of the mean lunar equator on the ecliptic is[44,45]

$$\begin{aligned} I &= 1°32'20'' = 1\overset{\circ}{.}53889 \\ &= 0.0268587 \text{ radian} \end{aligned} \tag{B-12}$$

Next, according to Ref. 46, we have

$$\begin{aligned} \Omega &= 259\overset{\circ}{.}183275 - 0\overset{\circ}{.}0529539222d \\ &\quad + 1\overset{\circ}{.}557 \times 10^{-12} d^2 + 5\overset{\circ}{.}0 \times 10^{-20} d^3 \\ ☾ - \Omega &= 11\overset{\circ}{.}250889 + 13\overset{\circ}{.}2293504490d \\ &\quad - 2\overset{\circ}{.}407 \times 10^{-12} d^2 - 1\overset{\circ}{.}1 \times 10^{-20} d^3 \end{aligned} \tag{B-13}$$

where d is the number of days that have elapsed from J.E.D. 2415020.0. Finally, the physical libration of the Moon is[47]

$$\begin{aligned} \tau = &-12\overset{''}{.}9 \sin \ell - 0\overset{''}{.}3 \sin 2\ell + 65\overset{''}{.}2 \sin \ell' \\ &+ 9\overset{''}{.}7 \sin(2F - 2\ell) + 1\overset{''}{.}4 \sin(2F - 2D) + 2\overset{''}{.}5 \sin(D - \ell) \\ &- 0\overset{''}{.}6 \sin(2D - 2\ell + \ell') - 7\overset{''}{.}3 \sin(2D - 2\ell) \\ &- 3\overset{''}{.}0 \sin(2D - \ell) - 0\overset{''}{.}4 \sin 2D + 7\overset{''}{.}6 \sin \Omega \quad ; \end{aligned} \tag{B-14}$$

$$\rho = -\,106'' \cos \ell + 35'' \cos(2F - \ell) - 11'' \cos 2F$$
$$- 3'' \cos(2F - 2D) - 2'' \cos(2D - \ell) \quad ; \tag{B-15}$$
$$I(\tau - \sigma) = 108'' \sin \ell - 35'' \sin(2F - \ell) + 11'' \sin 2F$$
$$+ 3'' \sin(2F - 2D) + 2'' \sin(2D - \ell) \tag{B-16}$$

where I is given by (B-12) measured in radians, where we have taken the parameter f in Ref. 47 to be f = 0.73, and where the arguments ℓ, ℓ', F and D are given in Ref. 40 as functions of time. The relations between the arguments ℓ, ℓ', F and D, and the arguments g, g', ω and ω' in Ref. 47 are given by[48]

$$\left.\begin{array}{ll} \ell = g & g = \ell \\ \ell' = g' & g' = \ell' \\ D = g' - g' + \omega - \omega' & \omega = F - \ell \\ F = g + \omega & \omega' = F - D - \ell' \end{array}\right\} . \tag{B-17}$$

APPENDIX C
ORIENTATION OF THE SUN

According to Ref. 49, we have

$$\left.\begin{array}{ll} \text{Inclination of solar equator to ecliptic} & I_s = 7°15' \\ \text{Longitude of ascending node of solar equator on ecliptic} & \Omega_s(t) = 73°40' + 50''.25t \end{array}\right\} \qquad \text{(C-1)}$$

where t is the time in years from 1850. Thus, in 1950.0, the longitude of the ascending node of the solar equator on the ecliptic was

$$\Omega_s = 75°3'.75 = 75°.0625 \quad . \qquad \text{(C-2)}$$

Now the rate of the precession of the equinox backward along the ecliptic is 50''.25 per year. Since formulas (C-1) were derived from observations, we can conclude that no precession of the solar equator along the ecliptic has been observed. If the Sun had an equatorial bulge, such a precession would arise from the gravitational action of the planets. Thus, the fact that no precession of the solar equator has been observed puts an upper limit on the possible magnitude of the second harmonic of the Sun's gravitational potential. However, the planetary torques acting on such a solar equatorial bulge would be so small that this upper limit is not much of a restriction.

Let (x^1, x^2, x^3) be the coordinate system with origin at the center of mass of the Sun whose x^3-axis points toward the north pole of the Sun, whose x^1-axis is the intersection of the equator of the Sun and the mean ecliptic of 1950.0, and whose x^2-axis completes the right-hand system. Let (u^1, u^2, u^3) be the coordinate system with origin at the center of mass of the Sun referred to the mean equinox and ecliptic of 1950.0. Then the results of Sec. II-A imply that

$$x^1 = u^1 \cos\Omega_s + u^2 \sin\Omega_s$$

$$x^2 = -u^1 \sin\Omega_s \cos I_s + u^2 \cos\Omega_s \cos I_s + u^3 \sin I_s$$

$$x^3 = u^1 \sin\Omega_s \sin I_s - u^2 \cos\Omega_s \sin I_s + u^3 \cos I_s \quad . \qquad \text{(C-3)}$$

Let (X^1, X^2, X^3) be the coordinate system with origin at the center of mass of the Sun referred to the mean equinox and equator of 1950.0. The relation between the (u^1, u^2, u^3) and (X^1, X^2, X^3) coordinate systems is given by (B-3), with $\epsilon_o = \bar{\epsilon}$ being the mean inclination of the ecliptic in 1950.0. Combining (C-3) and (B-3), we see that

$$x^1 = X^1 \cos\Omega_s + X^2 \sin\Omega_s \cos\bar{\epsilon} + X^3 \sin\Omega_s \sin\bar{\epsilon}$$

$$x^2 = -X^1 \sin\Omega_s \cos I_s + X^2(\cos\Omega_s \cos I_s \cos\bar{\epsilon} - \sin I_s \sin\bar{\epsilon})$$
$$+ X^3(\cos\Omega_s \cos I_s \sin\bar{\epsilon} + \sin I_s \cos\bar{\epsilon})$$

$$x^3 = X^1 \sin\Omega_s \sin I_s - X^2(\cos\Omega_s \sin I_s \cos\bar{\epsilon} + \cos I_s \sin\bar{\epsilon})$$
$$+ X^3(-\cos\Omega_s \sin I_s \sin\bar{\epsilon} + \cos I_s \cos\bar{\epsilon}) \quad . \qquad \text{(C-4)}$$

Finally, comparing (III-47) and (C-4), we see that the quantities C_{3k} (k = 1, 2, 3) appearing in (III-50) are

$$C_{31} = \sin \Omega_s \sin I_s$$

$$C_{32} = -\cos \Omega_s \sin I_s \cos \bar{\epsilon} - \cos I_s \sin \bar{\epsilon}$$

$$C_{33} = -\cos \Omega_s \sin I_s \sin \bar{\epsilon} + \cos I_s \cos \bar{\epsilon} \quad . \qquad \text{(C-5)}$$

REFERENCES

1. I.I. Shapiro, "Effects of General Relativity on Interplanetary Time-Delay Measurements," Technical Report 368, Lincoln Laboratory, M.I.T. (18 December 1964), DDC 614232.
2. D. Brouwer and G.M. Clemence, Methods of Celestial Mechanics (Academic Press, New York and London, 1961), pp. 31-33.
3. Ibid., pp. 1-25.
4. Ibid., pp. 47-49.
5. Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac (Her Majesty's Stationery Office, London, 1961), pp. 488-494.
6. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I (Interscience Publishers, New York, 1953), pp. 83-86.
7. C.W. Allen, Astrophysical Quantities, Second Edition (Athlone Press, University of London, 1963), p. 147.
8. Ibid., p. 141.
9. Ibid., p. 153.
10. R. Courant and D. Hilbert, op. cit., Vol. I, pp. 314 and 510-521.
11. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II (Interscience Publishers, New York, 1962), pp. 242-243.
12. C.W. Allen, op. cit., p. 108.
13. D. Brouwer and G.M. Clemence, op. cit., pp. 115-133.
14. C.W. Allen, op. cit., p. 149.
15. Ibid., p. 109.
16. J.L. Kelley, General Topology (Van Nostrand, New York, 1955).
17. S. Helgason, Differential Geometry and Symmetric Spaces (Academic Press, New York, 1962), Ch. I.
18. L.P. Eisenhart, Riemannian Geometry (Princeton University Press, New Jersey, 1926), p.17.
19. Ibid., pp. 19-21.
20. Ibid., pp. 48-50.
21. V. Fock, The Theory of Space Time and Gravitation (Pergamon Press, New York, 1959).
22. Ibid., pp. 168-171.
23. Ibid., pp. 179-184.
24. P.G. Bergmann, Introduction to the Theory of Relativity (Prentice-Hall, New York, 1942), pp. 198-203.
25. Ibid., p. 212.
26. G. de Rham, Variétés différentiables (Hermann, Paris, 1955), p. 129.
27. G.F.D. Duff, Partial Differential Equations (University of Toronto Press, Toronto, 1956), pp. 238-242.
28. V. Fock, op. cit., pp. 346-352.
29. Ibid., pp. 194-196.
30. R. Adler, M. Bazin and M. Schiffer, Introduction to General Relativity (McGraw-Hill, New York, 1965), pp. 174-177.

31. Ibid., pp. 177-187.

32. P.G. Bergmann, op. cit., pp. 212-217.

33. V. Fock, op. cit., pp. 196-199.

34. Ibid., pp. 215-217.

35. L. Infeld and J. Plebanski, Motion and Relativity (Pergamon Press, New York, 1960).

36. A. Einstein, L. Infeld and B. Hoffmann, "The Gravitational Equations and the Problem of Motion," Ann. Math. 39, 65 (1938).

37. The American Ephemeris and Nautical Almanac for the Year 1967 (U. S. Government Printing Office, Washington, D.C., 1965), p. 486. (Although the page number refers to this Ephemeris, it should be much the same for any year.)

38. Explanatory Supplement to the Ephemeris, op. cit., p. 31.

39. Ibid., p. 98.

40. Ibid., pp. 44-45.

41. Ibid., p. 43.

42. Ibid., p. 27.

43. K. Koziel, "Libration of the Moon" in Physics and Astronomy of the Moon, Z. Kopal, ed. (Academic Press, New York and London, 1962), p. 32.

44. Ibid., p. 57.

45. Explanatory Supplement to the Ephemeris, op. cit., p. 332.

46. Ibid., p. 107.

47. K. Koziel, op. cit., pp. 31-45.

48. Ibid., p. 34.

49. Explanatory Supplement to the Ephemeris, op. cit., p. 307.

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13. ABSTRACT

A computer program, called the Planetary Ephemeris Program (PEP), is being written at Lincoln Laboratory. The purpose of the program is to improve planetary and lunar ephemerides using the results of radar and optical observations. In this report, we derive the differential equations that are numerically integrated in the program to determine as functions of time the positions and velocities of the planets, of the Earth-Moon barycenter and of the Moon, and the partial derivatives of these positions and velocities with respect to initial conditions, masses and other parameters. Newtonian theory with the usual unrigorous general relativistic corrections is employed. The equations of motion and the equations for the partial derivatives with respect to initial conditions are presented in the form needed in the Encke's method of integration used in PEP.

14. KEY WORDS

planetary ephemerides	general relativity theory	computer applications
elliptic motion	gravitational potential	
Newtonian theory	Encke's equations	

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